A balanced independence number condition for a balanced bipartite graph to be Hamiltonian

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Abstract

Let $G$ be a 2-connected balanced bipartite graph with partite sets $X_1$ and $X_2$. We denote $\alpha_{BIP}(G)$ be the maximum cardinality of an independent set $A_1 \cup A_2$ of $G$ such that $A_1 \subset X_1$, $A_2 \subset X_2$ and $|A_1| - |A_2| \leq 1$. In this paper, we prove that if $\alpha_{BIP}(G) \leq 2\delta(G) - 2$, then $G$ is Hamiltonian. This condition is best possible, and this implies several known results, for example, in \cite{1, 6, 7, 11}. Also this theorem gives the positive answer to the problem posed by O. Favaron, P. Mago and O. Ordaz \cite{6} in 1993.

Keywords: Hamiltonian cycle, Bipartite graphs, Balanced independent set

AMS Subject Classification: 05C38, 05C45

1 Introduction

The order of the maximum independent set is called an \textit{independence number}. For a graph $G$, let $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ be the minimum degree, the independence number and the connectivity of $G$, respectively. An well-known Chvátal and Erdős theorem \cite{3} states that a 2-connected graph $G$ has a Hamiltonian cycle if $\alpha(G) \leq \kappa(G)$. Starting from this result, many researchers have extensively studied cycle-related properties of a graph with bounded independence number (comparing with the connectivity). See excellent surveys \cite{9, 13}.

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On the other hand, sometimes we consider Hamiltonian cycles in bipartite graphs, but unfortunately, it was shown that the problem of determining whether a given bipartite graph has a Hamiltonian cycle or not is NP-complete [10]. Thus, we consider sufficient conditions for the existence of a Hamiltonian cycle in a bipartite graph, and some results are known, for example, [4, 6, 11, 12]. See also Section 7.3 of the nice book [2] about bipartite graphs.

Combining these two directions of a study on Hamiltonian cycles, we have a natural question; How about the bipartite graphs case for the independence number condition? However, this question is not so interesting, because any bipartite graph $G$ has an independent set of order at least $|G|/2$.

By this reason, some researchers consider the balanced independence number of a bipartite graph instead of an ordinary independence number. For a bipartite graph $G$ with partite sets $X_1$ and $X_2$, let $\alpha_{BIP}(G)$ be the maximum cardinality of a balanced independent set of $G$, that is,

$$\alpha_{BIP}(G) := \max \{ |A_1 \cup A_2| \mid A_1 \cup A_2 \text{ is an independent set of } G \}
\text{ with } A_1 \subset X_1, A_2 \subset X_2 \text{ and } |A_1| = |A_2|.$$ 

Ash [1] proved that a 2-connected bipartite graph $G$ with $\delta(G) \geq (|G|+1)/6$ and $\alpha_{BIP}(G) < (2\delta(G) - 3)/3$ is Hamiltonian. Fraisse [7] improved this result; a 2-connected balanced bipartite graph $G$ with $\alpha_{BIP}(G) \leq \delta(G) - 1$ is Hamiltonian. Favaron, Mago, Maulino and Ordaz [5] proved that a balanced bipartite graph $G$ with $\alpha_{BIP}(G) \leq 2 \leq \delta(G)$ but having no Hamiltonian cycle is isomorphic to $\Theta$ or $\Theta_1$ (where $\Theta$ and $\Theta_1$ are the graphs shown in Figure 1). In [6], Favaron Mago and Ordaz improved these results and showed the following theorem.

**Theorem 1.1 (Favaron, Mago and Ordaz [6])** Let $G$ be a balanced bipartite graph. If $\alpha_{BIP}(G) \leq \delta(G)$, then $G$ is Hamiltonian, or isomorphic to $\Theta$ or $\Theta_1$.

However, all the conditions on $\alpha_{BIP}(G)$ as mentioned above are not best possible for graphs with $\delta(G) \geq 5$. Actually, in [6], Favaron et al. suggested that the weaker condition $\alpha_{BIP}(G) \leq
2\delta(G) - 4 might be able to force the Hamiltonicity for a balanced bipartite graph G. In this paper, we show that their suggestion is true.

**Theorem 1.2** Let G be a 2-connected balanced bipartite graph. If \( \alpha_{BIP}(G) \leq 2\delta(G) - 4 \), then G is Hamiltonian.

Here we prove a stronger result than Theorem 1.2. In the following result, we consider an “almost balanced” independent set with one difference allowed. For a bipartite graph G with partite sets \( X_1 \) and \( X_2 \), let \( \ast BIP(G) := \max \{|A_1 \cup A_2| \mid A_1 \cup A_2 \text{ is an independent set of } G \}

\text{with } A_1 \subset X_1, A_2 \subset X_2 \text{ and } |A_1| - |A_2| \leq 1 \} \).

**Theorem 1.3** Let G be a 2-connected balanced bipartite graph. If \( \ast BIP(G) \leq 2\delta(G) - 2 \), then G is Hamiltonian.

**Remark 1.4** The condition on Theorem 1.2 implies that on Theorem 1.3. That is why, if G has an independent set \( A_1 \cup A_2 \) with \( A_1 \subset X_1, A_2 \subset X_2, |A_1| - |A_2| \leq 1 \) and \( |A_1 \cup A_2| \geq 2\delta(G) - 1 \), then \( A_i \) must contain a subset \( A'_i \) with \( |A'_i| = \delta(G) - 1 \) for \( i = 1, 2 \), and hence \( A'_1 \cup A'_2 \) is a balanced independent set of order \( 2\delta(G) - 2 \). Thus, if \( \alpha_{BIP}(G) \leq 2\delta(G) - 4 \), then \( \ast BIP(G) \leq 2\delta(G) - 2 \).

**Remark 1.5** If \( \delta(G) \geq 2 \), then the condition “2-connected” on Theorem 1.3 is not necessary. Let G be a balanced bipartite graph such that \( \delta(G) \geq 2 \) and \( \ast BIP(G) \leq 2\delta(G) - 2 \). Note that \( \alpha_{BIP}(G) \leq \ast BIP(G) \). As is shown in [6], if \( \alpha_{BIP}(G) \leq \delta(G) \), then \( \kappa(G) = \delta(G) \geq 2 \), and if \( \delta(G) + 1 \leq \alpha_{BIP}(G) \leq 2\delta(G) - 2 \), then \( \kappa(G) \geq 2\delta(G) - \alpha_{BIP}(G) \geq 2 \).

**Remark 1.6** Lu [11] showed that a balanced bipartite graph G with \( \ast BIP(G) \leq |G| \cdot \lambda(G) - 1 \) is Hamiltonian, where \( \lambda(G) \) is the invariant concerning the edge density of G. As is mentioned in [11], it is obvious that \( |G| \cdot \lambda(G) \leq \delta(G) \), and hence the result on [11] is a direct corollary of Theorem 1.3.

We show two examples that states the balanced independence number conditions of Theorems 1.2 and 1.3 are best possible. The first one was introduced in [6].

**Example 1:** Let \( G_1 \) be the graph obtained from three disjoint copies of \( K_{\delta-1, \delta-1} \) by joining a new vertex \( a \) to all the vertices of each of one partite set and a new vertex \( b \) to all the vertices of other partite set. Then \( G_1 \) is not Hamiltonian since removing the two vertices \( a \) and \( b \) makes
three components. Also \( \alpha_{BIP}(G_1) = 2\delta - 2 = 2\delta(G_1) - 2 \), and \( \alpha_{BIP}^*(G_1) = 2\delta - 1 = 2\delta(G_1) - 1 \).

**Example 2:** Let \( Y_1, Y_2, Z_1 \) and \( Z_2 \) be sets of vertices with \(|Y_1| = |Y_2| = m \) and \(|Z_1| = |Z_2| = 2m + 1 \) for an integer \( m \geq 1 \). We construct the graph \( G_2 \) by joining \( Y_1 \) and \( Y_2 \), \( Y_2 \) and \( Z_1 \), and \( Y_1 \) and \( Z_2 \), respectively, and adding \( 2m + 1 \) edges so that they form a matching between \( Z_1 \) and \( Z_2 \). Note that \( G_2 \) is a bipartite graph with partite sets \( Y_1 \cup Z_1 \) and \( Y_2 \cup Z_2 \). Notice also that \( \delta(G_2) = m + 1 \). If we remove all vertices in \( Y_1 \cup Y_2 \), (note that \(|Y_1 \cup Y_2| = 2m, \)) then the resultant graph is only a matching with \( 2m + 1 \) edges, and hence \( G_2 \) is not Hamiltonian. Moreover, for any almost balanced independent set \( A_1 \cup A_2 \) of \( G_2 \) of order at least two with \( A_i \subset Y_i \cup Z_i \), we have \( A_i \cap Y_i = \emptyset \), that is, \( A_i \subset Z_i \). Since we can choose only one vertex from any edge in the matching between \( Z_1 \) and \( Z_2 \) as an independent set, we have \(|A_1| + |A_2| \leq 2m + 1 = 2\delta(G_2) - 1 \). Therefore, the upper bound of the condition of Theorem 1.3 is best possible.

Finally, we mention a Chvátal-Erdős type condition for bipartite graphs. Since \( \kappa(G) \leq \delta(G) \), we can obtain the following as a corollary of Theorem 1.3. Note that the condition of Corollary 1.7 is best possible since the graph \( G_2 \) in Example 2 satisfies \( \kappa(G_2) = \delta(G_2) \).

**Corollary 1.7** Let \( G \) be a 2-connected balanced bipartite graph. If \( \alpha_{BIP}^*(G) \leq 2\kappa(G) - 2 \), then \( G \) is Hamiltonian.

## 2 Preliminaries

We first prove the following lemma. For a tree \( T \), let \( L(T) \) be the set of leaves of \( T \). (For convenience, when \(|T| = 1 \) and \(|T| = 2 \), we define \( L(T) \) as an emptyset and a set consisting of only one vertex of \( T \), respectively.) A tree \( T \) is said to be a caterpillar if \( T - L(T) \) is a path, called the spine of \( T \).

**Lemma 2.1** Let \( T \) be a forest such that each component of \( T \) is a caterpillar, and let \( X_1 \) and \( X_2 \) be partite sets of \( T \). Then there exist \( A_1 \subset X_1 \) and \( A_2 \subset X_2 \) such that \( A_1 \cup A_2 \) is an independent set of \( T \), and for some \( i = 1 \) or \( 2 \), \(|A_i| \geq \lceil \frac{|X_i|}{2} \rceil \) and \(|A_{3-i}| \geq \lceil \frac{|X_{3-i}|}{2} \rceil \).

**Proof.** We prove this lemma by the induction on \(|T| \). When \(|T| \leq 3 \), we can easily take the desired independent set. Thus, we may assume that \(|T| \geq 4 \).

First we show that we may assume that \( T \) is connected, that is, \( T \) is a tree. Suppose that \( T \) is not connected. If all components of \( T \) have no edges, then we can easily obtain the desired independent set. Hence we may assume that there exist two components \( T_1 \) and \( T_2 \) of \( T \) such
that $|T_2| \geq 2$. We can choose an end vertex of the spine of $T_1$ and an end vertex of the spine of $T_2$ or a leaf incident with an end vertex of the spine of $T_2$ so that the two vertices are contained in distinct partite sets of $T$, and add a new edge between them in order to connect $T_1$ and $T_2$. By this operation, we can keep the property that each component is a caterpillar, and any independent set of the obtained graph is also independent in the original graph. Hence repeating this operation until $T$ is connected, we may assume that $T$ is connected.

**Case 1.** $L(T) \cap X_1 \neq \emptyset$ and $L(T) \cap X_2 \neq \emptyset$.

Then there exist $x_1 \in L(T) \cap X_1$ and $x_2 \in L(T) \cap X_2$. Let $y_1$ and $y_2$ be vertices such that $x_1y_2, y_1x_2 \in E(T)$. Let $T' := T - \{x_1, y_1, x_2, y_2\}$, $X_1' := X_1 - \{x_1, y_1\}$ and $X_2' := X_2 - \{x_2, y_2\}$. Then $T'$ is a forest with $|T'| = |T| - 4$. Hence by the induction hypothesis, there exist $A_1' \subset X_1'$ and $A_2' \subset X_2'$ such that $A_1' \cup A_2'$ is an independent set of $T'$, and for some $i = 1$ or $2$, $|A_1'| \geq \lceil \frac{|X_1'|}{2} \rceil$ and $|A_{3-i}'| \geq \lfloor \frac{|X_{3-i}'|}{2} \rfloor$. We may assume that $i = 1$. Let $A_1 := A_1' \cup \{x_1\}$ and $A_2 := A_2' \cup \{x_2\}$. Then $A_1 \cup A_2$ is an independent set of $T$ with $|A_1| \geq \lceil \frac{|X_1|}{2} \rceil + 1 = \lceil \frac{|X_1|}{2} \rceil$ and $|A_2| \geq \lceil \frac{|X_2|}{2} \rceil + 1 = \lceil \frac{|X_2|}{2} \rceil$.

**Case 2.** $L(T) \subset X_1$ or $L(T) \subset X_2$.

By the symmetry of $X_1$ and $X_2$, we may assume that $L(T) \subset X_1$. Let $P$ be the spine of $T$. Since $L(T) \subset X_1$, $X_2 \subset V(P)$. Note that $|P|$ is odd because both end vertices of $P$ are contained in $X_2$. So, we can take the center vertex $x$ of $P$. If $|P| \geq 3$, then let $T^d$ and $T^v$ be two components of $T - x$ such that $V(P) \cap V(T^d) \neq \emptyset$ and $V(P) \cap V(T^v) \neq \emptyset$; otherwise, let $T^d := \emptyset$ and $T^v := \emptyset$. Note that $|V(T^d) \cap X_2| = |V(T^v) \cap X_2|$. By the symmetry of $T^d$ and $T^v$, we may assume that $|V(T^d) \cap X_1| \geq |V(T^v) \cap X_1|$. Let $L_1 := V(T - T^d - T^v - x)$, and let $A_1 := (V(T^d) \cap X_1) \cup L_1$ and $A_2 := V(T^v) \cap X_2$. By the choice of $A_1$ and $A_2$, $A_1 \cup A_2$ is an independent set of $T$ with

$$|A_1| \geq \lceil \frac{|X_1|}{2} \rceil \quad \text{and} \quad |A_2| = \lceil \frac{|X_2|}{2} \rceil \quad \text{if } x \in X_1,$$

$$|A_1| \geq \lfloor \frac{|X_1|}{2} \rfloor \quad \text{and} \quad |A_2| = \lfloor \frac{|X_2|}{2} \rfloor \quad \text{if } x \in X_2. \quad \blacksquare$$

We also use the following result concerning a length of the longest cycle.

**Theorem 2.2 (Jackson [8])** Let $G$ be a 2-connected balanced bipartite graph. Then $G$ has a cycle of length at least $4\delta(G) - 2$, or $G$ is Hamiltonian.

Our notation is standard possibly except the following. Let $G$ be a graph. We denote by $N_G(x)$ the neighborhood of a vertex $x$ in $G$. For a subgraph $H$ of $G$ and a vertex $x \in V(G) - V(H)$,
we denote $N_H(x) := N_G(x) \cap V(H)$, and $d_H(x) := |N_H(x)|$. If there is no possible confusion, we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$. We write a cycle (or a path) $C$ with a given orientation by $\overrightarrow{C}$. If there exists no fear of confusion, we abbreviate $\overrightarrow{C}$ by $C$. Let $\overrightarrow{C}$ be an oriented cycle or a path. For $x, y \in V(C)$, we denote by $x \overrightarrow{C} y$ a path from $x$ to $y$ on $\overrightarrow{C}$. The reverse sequence of $x \overrightarrow{C} y$ is denoted by $y \overleftarrow{C} x$. For $u \in V(C)$, we denote the $h$-th successor and the $h$-th predecessor of $u$ on $\overrightarrow{C}$ by $u^+$ and $u^-$, respectively. We abbreviate $u^{+1}$ and $u^{-1}$ by $u^+$ and $u^-$, respectively. For $X \subset V(C)$, we define $X^+ := \{x^+ | x \in X\}$. For a matching $M$, we let $V(M)$ be the set of vertices which is an end vertex of some edge in $M$.

3 Proof of Theorem 1.3

Let $G$ be a 2-connected balanced bipartite graph with partite sets $X_1$ and $X_2$ satisfying the conditions of Theorem 1.3. Assume that $G$ does not have a Hamiltonian cycle.

If $\delta(G) = 2$, then $\alpha_{BIP}(G) \leq 2$ by the conditions of Theorem 1.3. This implies that $\alpha_B(G) \leq 2$, and hence by Theorem 1.1, $G$ is isomorphic to $\Theta$ or $\Theta_1$. (See Figure 1.) However, $\alpha_{BIP}(\Theta) = \alpha_{BIP}(\Theta_1) = 3$, a contradiction. Therefore $\delta(G) \geq 3$.

For a cycle $C$ of $G$, let

$$g(C) := \min \{8, \max \{|V(M)| | M \text{ is a matching of } G - C\}\}.$$ 

Choose a cycle $C$ of $G$ so that

(C1) $|C| + g(C)$ is as large as possible, and

(C2) $|C|$ is as large as possible, subject to (C1).

We give an orientation to the cycle $C$. Let $H := G - C$. We divide the proof into two cases.

3.1 The case $g(C) = 0$

For $v \in V(C)$, let $U(v) := \{v, v^+\}$. For $i = 1, 2$, let $x_i \in V(H) \cap X_i$, and let

$$S_i := (N_C(x_i)^+ \cap N_C(x_i)^-) \cup \{x_i\},$$

$$T_i := N_C(x_i)^+ - S_i,$$

and

$$L_i := \{v \in T_i | U(v) \cap N_G(u) \neq \emptyset \text{ for some } u \in S_{3-i}\}.$$

Claim 1  

(i) $S_1 \cup S_2$ is an independent set of $G$.

(ii) $|L_i| \leq 1$ for $i = 1, 2$.  

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Claim 2

Proof. (i) Suppose that there exist \( u \in S_1 \) and \( v \in S_2 \) such that \( uv \in E(G) \) and \( uv \not\in E(C) \), then by taking a cycle \( C' := x_1u^+v^-x_2v^+u^-x_1 \) contradicting (C1), since \( |C'| = |C| \) and \( g(C') \geq 2 \). In other cases, we can also get a contradiction by the similar argument.

(ii) Suppose that there exist \( u_1, v_1 \in L_1 \) with \( u_1 \neq v_1 \). Let \( u_2, v_2 \in S_2 \) be vertices such that \( U(u_1) \cap N_C(u_2) \neq \emptyset \) and \( U(v_1) \cap N_C(v_2) \neq \emptyset \). Then we can take \( u'_1 \in U(u_1) \) and \( v'_1 \in U(v_1) \) so that \( u'_1u_2, v'_1v_2 \in E(G) \). Recall that \( u'_1 = u_1 \) or \( v_1^{i_2} \) and \( v'_1 = v_1 \) or \( u_1^{i_2} \). By symmetry, we may assume that \( u'_2 \neq x_2 \) if \( u_2 \neq x_2 \) and \( u_1 \in V(u'_2v'_1) \) if \( u_2 \in V(C) \). Depending on the case on \( u_2, v_2 \) and on the order of \( u_1, v_1, u_2, v_2 \), we let the cycle \( C' \) as in Table 1. Note that \( (V(C) - \{u_1, u_1^{i_2}, v_1, v_1^{i_2}\}) \cup \{x_1, x_2\} \subset V(C') \). Let \( t := |E(C) - E(C')| \). Then \( g(C') \geq 2l = g(C) + 2l \) because \( 2l \leq 4 \), and hence \( |C'| + g(C') = (|C| + 2 - 2l) + g(C) + 2l > |C| + g(C) \), contradicting (C1).

<table>
<thead>
<tr>
<th>( C' )</th>
<th>cases</th>
<th>order in ( \overline{G} )</th>
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<tbody>
<tr>
<td>( u_2u'_1v'u'_2v_1x_1u_1v'v_2x_2 )</td>
<td>( u_2 = v_2 = x_2 )</td>
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<td>( x_2u'_2v'u'_1v_2x_1u_1v'v_2x_2 )</td>
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<td>( u_1, v_2, v_1 )</td>
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Table 1: The cycle \( C' \) in the proof of Claim 1 (ii).

Let \( U_i := \{U(v) \mid v \in T_i-L_i\} \) for \( i = 1, 2 \). We define the bipartite graph \( R \) with partite sets \( U_1 \) and \( U_2 \) as follows:

\[
V(R) := U_1 \cup U_2, \\
E(R) := \{U(v_1)U(v_2) \mid \{v_1v_2, v_1v_2^{i_2}, v_1v_2^{i_2}, v_1v_2^{i_2}v_1v_2^{i_2}\} \cap E(G) \neq \emptyset\}.
\]

Then \( R \) has a restricted property as follows.

Claim 2 \( R \) is a forest such that each component is a caterpillar.
Proof. First we consider a property of two edges of $R$. Let $U(u_1)U(u_2), U(v_1)U(v_2)$ be two edges of $R$ with $U(u_i), U(v_i) \in \mathcal{U}_i$ and $u_i \neq v_i$ for $i = 1, 2$. By the symmetry of $\mathcal{U}_1$ and $\mathcal{U}_2$, we may assume that $u_1, u_2, v_1, v_2$ appear in $\overrightarrow{C}$ in the order (a) $u_1, u_2, v_1, v_2$ or (b) $u_1, v_1, u_2, v_2$ or (c) $u_1, v_1, v_2, u_2$. By the definition of $R$, we can take $u'_i \in U(u_i)$ and $v'_i \in U(v_i)$ $(i = 1, 2)$ so that

$u'_1u'_2, v'_1v'_2 \in E(G)$. For the case (a) or (b), we consider the following cycle $C'$. Let

$$C' := \begin{cases} x_1u'_1 \overrightarrow{C} v'_2v'_1 \overrightarrow{C} v'_2 x_2u'_2 \overrightarrow{C} u'_1u'_2 \overrightarrow{C} v'_1x_1 & \text{in the case (a)}, \\
 x_1u'_1 \overrightarrow{C} v'_2v'_1 \overrightarrow{C} u'_2x_2v'_2 \overrightarrow{C} u'_1u'_2 \overrightarrow{C} v'_1x_1 & \text{in the case (b)}. \end{cases}$$

Let $l := \{|u_1u_1^+, u_2u_2^+, v_1v_1^+, v_2v_2^+\} - E(C')$. Then $|C'| = |C| + 2 - 2l$ and $G - C'$ has a matching with at least $2l$ edges. Since $2l \leq 8$, we obtain a contradiction as in the proof of Claim 1 (ii). So neither the case (a) nor (b) occur, and hence $u_1, u_2, v_1, v_2$ appear in $\overrightarrow{C}$ in the order (c) $u_1, v_1, v_2, u_2$.

Using this fact, for a path of $R$ of order four, we have the following property.

$$(3.1) \text{For any path } U(u)U(v)U(w)U(y) \text{ of } R \text{ of order four, } y \in V(x^+\overrightarrow{C} w^-) \text{ if } u, v, w \text{ occur in this order along } \overrightarrow{C}; \text{ otherwise, } y \in V(w^+\overrightarrow{C} x^-).$$

By $(3.1)$, we can easily see that $R$ has no cycle. Let $U(x)$ be a vertex of $R$ of degree at least three, and let $U(u), U(v), U(w)$ be distinct neighbors of $U(x)$ in $R$. To show that each component of $R$ is a caterpillar, we have only to prove that $d_R(U(z)) = 1$ for some $z \in \{u, v, w\}$. We may assume that $u, x, v, w$ are appear in this order along $\overrightarrow{C}$. Suppose that $d_R(U(w)) \geq 2$, and let $U(y) \in N_R(U(w))$ with $y \neq x$. By applying $(3.1)$ to $u, x, w, y$ and $v, x, w, y$ respectively, we obtain $y \in V(x^+\overrightarrow{C} w^-)$ and $y \in V(w^+\overrightarrow{C} x^-)$, a contradiction. 

By Lemma 2.1, Claim 2 and the symmetry of $\mathcal{U}_1$ and $\mathcal{U}_2$, we may assume that there exist $\mathcal{W}_1 \subseteq \mathcal{U}_1$ and $\mathcal{W}_2 \subseteq \mathcal{U}_2$ such that $\mathcal{W}_1 \cup \mathcal{W}_2$ is an independent set of $R$, $|\mathcal{W}_1| \geq \lceil \frac{|\mathcal{U}_1|}{2} \rceil$ and $|\mathcal{W}_2| \geq \lceil \frac{|\mathcal{U}_2|}{2} \rceil$. For $i = 1, 2$, let

$$A_i := S_i \cup \bigcup_{U(v) \in \mathcal{W}_i} U(v).$$

By Claim 1 (i) and the definitions of $S_i$, $\mathcal{U}_i$ and $\mathcal{W}_i$, $A_1 \cup A_2$ is an independent set of $G$. By Claim 1 (ii),

$$|A_1| = |S_1| + \sum_{U(v) \in \mathcal{W}_i} |U(v)| = |S_1| + 2|\mathcal{W}_1| \geq |S_1| + 2\left\lfloor \frac{|\mathcal{U}_1|}{2} \right\rfloor \geq |S_1| + |\mathcal{W}_1| = |S_1| + |T_1| - |L_1| \geq (d_C(x_1) + 1) - 1 \geq \delta(G),$$

and

$$|A_2| \geq |S_2| + 2\left\lfloor \frac{|\mathcal{U}_2|}{2} \right\rfloor \geq |S_2| + |\mathcal{U}_2| - 1 \geq d_C(x_2) - 1 \geq \delta(G) - 1,$$
a contradiction.

3.2 The case $g(C) \geq 2$

By Theorem 2.2, there exists a cycle $C'$ of $G$ with $|C'| \geq 4\delta(G) - 2$. By the choice (C1), $|C| + g(C) \geq |C'| + g(C') \geq 4\delta(G) - 2$. If $|C| + g(C) = 4\delta(G) - 2$, then $|C| + g(C) = |C'| + g(C')$ and $g(C') = 0$, and this implies that $|C| < |C'|$ because $g(C) \geq 2$, contradicting the choice (C2). Therefore, we have

$$|C| \geq 4\delta(G) - g(C) \geq 4\delta(G) - 8. \quad (3.2)$$

For a maximum matching $M$ of $H$, an oriented path $\overrightarrow{P} = v_1v_2v_3\ldots$ is called $M$-alternating if $v_1v_2 \in M$, $N_C(x_1) \neq \emptyset$ and $\overrightarrow{P}$ passes through edges in $M$ and not in $M$, alternatively. (Usually an $M$-alternating path does not necessarily satisfy the first and second conditions, but for convenience, here we require them in the definition.) Note that there exist a maximum matching $M$ of $H$ and an $M$-alternating path $\overrightarrow{P} = x_1x_2\ldots x_{2p+\varepsilon}$, where $\varepsilon = 0$ or $\varepsilon = 1$. We need the following claim.

**Claim 3** Suppose that $\varepsilon = 0$ or suppose that $\overrightarrow{P}$ is a longest $M$-alternating path. Let $u \in N_C(x_1)^+$ and $v \in N_C(x_{2p+\varepsilon})^+$ with $u \neq v$. If $uv^{+i} \in E(G)$, then $i \geq |P|$. In particular, $|v\overrightarrow{C}u^{-2}| \geq |P|$.

**Proof.** Suppose that $i \leq |P| - 1$. If $v^{+i} \notin V(v^+\overrightarrow{C}u^-)$, then $u^- = v^{+j}$ for some $j < i$. Since $uu^- \in E(G)$, we can use $u^- = v^{+j}$ instead of $v^{+i}$. Renaming $j$ to $i$, we may assume that $v^{+i} \in V(v^+\overrightarrow{C}u^-)$. Let

$$C' := u^-x_1\overrightarrow{P}_{x_{2p+\varepsilon}}v^-Cuv^{+i}\overrightarrow{C}u^-,$$

and $M' := \begin{cases} M - (M \cap E(P)) \cup \{v^{+}, v^{+2}, v^{+3}, \ldots, v^{+(i-3)}v^{+(i-2)}\} & \text{if } i \text{ is odd}, \\ M - (M \cap E(P)) \cup \{v^{+}, v^{+2}, v^{+3}, \ldots, v^{+(i-2)}v^{+(i-1)}\} & \text{otherwise.} \end{cases}$

Then $C'$ is a cycle of $G$ with $|C'| = |C| - i + |P| > |C|$, and by the assumption of Claim 3, $M'$ is a matching of $G - C'$ and $|V(M')| = |V(M)| - |P| + i$. Then $g(C') \geq \min\{8, |V(M)| - |P| + i\}$, and hence it follows from the equality $g(C) = \min\{8, |V(M)|\}$ that

$$|C'| + g(C') \geq (|C| - i + |P|) + \min\{8, |V(M)| - |P| + i\}$$

$$= |C| + \min\{8 - i + |P|, |V(M)|\}$$

$$\geq |C| + g(C).$$

9
Choose a maximum matching $M$ of $H$ and an $M$-alternating path $\overrightarrow{P}$ in $H$ so that

(M1) $\overrightarrow{P}$ is as long as possible, and

(M2) we can take a matching with two edges between $\{x_1, x_{2p+\varepsilon}\}$ and $V(C)$ if possible, subject to (M1).

By the symmetry of $X_1$ and $X_2$, we may assume that $x_1 \in X_1$. By the choices (M1) and (M2), we have the following claims.

**Claim 4** For any $x \in N_P(x_{2p+\varepsilon})^+$, the following hold.

(i) There exist a maximum matching $M'$ of $H$ and an $M'$-alternating path $\overrightarrow{Q}$ from $x_1$ to $x$ such that $V(Q) = V(P)$.

(ii) $N_H(x) \subset V(P) \cap X_{1+\varepsilon}$.

**Proof.** (i) The following are the desired matching $M'$ and $M'$-alternating path $\overrightarrow{Q}$. Note that $\Delta$ means the symmetric difference between two sets.

\[
\overrightarrow{Q} := x_1 \overrightarrow{P} x_{2p+\varepsilon} \overrightarrow{P} x,
\]

and $M' := \begin{cases} M \Delta E(x \overrightarrow{P} x_{2p+1}) & \text{if } \varepsilon = 1, \text{ that is, } x_{2p+1} \in X_1, \\ M \Delta E(x \overrightarrow{P} x_{2p}) \Delta \{x_{2p+1}\} & \text{if } \varepsilon = 0, \text{ that is, } x_{2p} \in X_2. \end{cases}$

(ii) Let $M'$ be a maximum matching of $H$ and $\overrightarrow{Q}$ be an $M'$-alternating path as in (i). Suppose that there exists $z \in N_H(x) - V(Q)$. If $\varepsilon = 0$, then $\overrightarrow{Q}xz$ is an $M'$-alternating path which is longer than $\overrightarrow{P}$, contradicting (M1). Thus $\varepsilon = 1$. If $z \in V(M')$, then by letting $zz' \in M'$, $\overrightarrow{Q}xz$ is an $(M' - \{zz'\}) \cup \{xz\}$-alternating path which is longer than $\overrightarrow{P}$, a contradiction. Thus $z \notin V(M')$. Then $M' \cup \{zx\}$ is a matching in $H$, contradicting the maximality of $M$. Therefore we have $N_H(x) \subset V(Q)$. Since $V(Q) = V(P)$ and $x \in X_{2-\varepsilon}$, $N_H(x) \subset V(P) \cap X_{1+\varepsilon}$. ■

**Claim 5** There exists a matching with two edges between $\{x_1, x_{2p+\varepsilon}\}$ and $V(C)$.

**Proof.** Suppose not, and let $u \in N_C(x_1)^+$. By Claim 4 (i) and the choice (M2), $N_C(x_1)^+ \subset \{u\}$ for any $x \in N_P(x_{2p+\varepsilon})^+$. Especially, if $\varepsilon = 0$, then $N_C(x) = \emptyset$ for any $x \in N_P(x_{2p+\varepsilon})^+$ because $N_P(x_2p)^+ \subset X_2$ and $u \in X_1$. Note that if $N_C(x_{2p+\varepsilon}) \neq \emptyset$, then $N_C(x_1)^+ = \{u\}$. Since $H$ is
balanced and \(|V(P) \cap X_1| = |V(P) \cap X_2| + \varepsilon\), there exists a vertex \(z \in (V(H) \cap X_2) - V(P)\) if \(\varepsilon = 1\). Let

\[
A_1 := \begin{cases} 
N_P(x_{2p+\varepsilon})^+ & \text{if } N_C(x_{2p+\varepsilon}) = \emptyset, \\
N_P(x_{2p+\varepsilon})^+ \cup \{x_1\} & \text{if } N_C(x_{2p+\varepsilon}) \neq \emptyset,
\end{cases}
\]

and \(A_2 := \begin{cases} 
(V(C) \cap X_1) & \text{if } \varepsilon = 0, \\
(V(C) \cap X_2 - \{u^-\}) \cup \{z\} & \text{if } \varepsilon = 1.
\end{cases}\)

Then \(A_1 \cup A_2\) is an independent set of \(G\) with \(A_1 \subset X_{2-\varepsilon}\) and \(A_2 \subset X_{1+\varepsilon}\). By Claim 4 (ii), \(|A_1| = d_P(x_{2p+\varepsilon}) = d_G(x_{2p+\varepsilon}) \geq \delta(G)\) or \(|A_1| = d_P(x_{2p+\varepsilon}) + 1 = d_G(x_{2p+\varepsilon}) \geq \delta(G)\) according as \(N_C(x_{2p+\varepsilon}) = \emptyset\) or \(N_C(x_{2p+\varepsilon}) \neq \emptyset\), and hence \(|V(C) \cap X_{1+\varepsilon}| = |A_2| \leq \delta(G) - 2\). By the equality \((3.2), 4\delta(G) - 8 \leq |C| = 2|V(C) \cap X_{1+\varepsilon}| \leq 2\delta(G) - 4, or \delta(G) \leq 2, a contradiction. \(\blacksquare\)

By Claim 5, there exist \(u \in N_C(x_1)^+\) and \(v \in N_C(x_{2p+\varepsilon})^+\) with \(u \neq v\). We choose such vertices \(u\) and \(v\) so that \(|\overrightarrow{vC}u|\) is as small as possible. Then \(N_C(x_{2p+\varepsilon}) \cap V(\overrightarrow{vC}u^2) = \emptyset\). Note that \(|\overrightarrow{vC}u^2| \geq |P|\) by Claim 3. Hereafter, we use the symmetry of \(x_1\) and \(x_{2p+\varepsilon}\) because \(\overrightarrow{P}\) is an \(M\Delta E(P)\)-alternating path (if \(\varepsilon = 1\)) or an \(M\)-alternating path (if \(\varepsilon = 0\)) satisfying the choice (M1) and (M2). We divide the rest of the proof into two parts depending on the value of \(\varepsilon\).

**Case 1.** \(\varepsilon = 0\).

Let \(A_1 := N_C(x_1)^+ \cup N_P(x_1)^-\),

and \(A_2 := N_C(x_{2p})^+ \cup \{v^{+2}, v^{+4}, \ldots, v^{+(2p-2)}\}\).

By Claims 3, 4 and the symmetry of \(x_1\) and \(x_{2p}\), \(A_1 \cup A_2\) is an independent set of \(G\). By the choice of \(u\) and \(v\), \(N_C(x_{2p})^+ \cap \{v^{+2}, v^{+4}, \ldots, v^{+(2p-2)}\} = \emptyset\). It follows from Claim 4 (ii) and the symmetry of \(x_1\) and \(x_{2p}\) that \(|A_1| = d_C(x_1) + d_P(x_1) = d_C(x_1) + d_H(x_1) \geq \delta(G)\) and \(|A_2| = d_C(x_{2p}) + (p - 1) = d_C(x_{2p}) + |V(P) \cap X_1| - 1 \geq d_C(x_{2p}) + d_H(x_{2p}) - 1 \geq \delta(G) - 1\), a contradiction. \(\blacksquare\)

**Case 2.** \(\varepsilon = 1\).

Let \(K_1 := N_H(x_2) - V(P)\),

and \(J_2 := (V(H) \cap X_2) - V(P)\).

Since \(H\) is balanced and \(|V(P) \cap X_1| = |V(P) \cap X_2| + 1\), we obtain that \(|J_2| = |V(H) \cap X_2| - |V(P) \cap X_2| = |V(H) \cap X_1| - (|V(P) \cap X_1| - 1) \geq |K_1| + 1\). Thus, we can take \(J_2' \subset J_2\) with

\[
|J_2'| = |K_1| + 1. \quad (3.3)
\]
Let

\[ I_1 := N_C(x_1)^+ - \bigcup_{z \in J'_2} N_C(z), \]
\[ A_1 := K_1 \cup I_1 \cup N_P(x_1)^-, \]
and \[ A_2 := J'_2 \cup \{v^+, v^{+3}, \ldots, v^{+(2p-1)}\} \cup N_C(x_2)^+. \]

We shall show that \( A_1 \cup A_2 \) is an independent set of \( G \) with \( A_1 \subset X_1, A_2 \subset X_2, |A_1| \geq \delta(G) - 1 \) and \( |A_2| \geq \delta(G) \).

For \( z \in K_1 \cup N_P(x_1)^- \), we define a matching \( M' \) of \( H \) and an \( M' \)-alternating path \( \overrightarrow{Q} \) as follows. If \( z \in K_1 \), then let \( Q := x_{2p+1}^+ \overrightarrow{P} x_{2z}^+ \) and \( M' := M \Delta E(P) \); if \( z \in N_P(x_1)^- \), then let \( Q := x_{2p+1}^+ \overrightarrow{P} z \overrightarrow{P} x_{2p+1}^+ \) and \( M' := M \Delta E(z \overrightarrow{P} x_{2p+1}) \). Then \( M' \) is a maximum matching of \( H \), and \( \overrightarrow{Q} \) is an \( M' \)-alternating path from \( x_{2p+1} \) to \( z \) with \( |Q| = |P| \). Hence by the choice \((M1), N_H(z) \subset V(Q) \cap X_2 = V(P) \cap X_2 \). Hence \( K_1 \cup N_P(x_1)^- \cup J_2 \) is independent. Since \( I_1 \cup J'_2 \) is independent by the definition of \( I_1 \), \( A_1 \cup J'_2 \) is independent. In particular, since \( N_H(x_1) \cap J_2 = \emptyset \),

\[ d_H(x_1) = d_P(x_1). \] (3.4)

Moreover, by Claim 3, \( A_1 \cup \{v^+, v^{+3}, \ldots, v^{+(2p-1)}\} \) is independent. Note that \( x_1 x_2 \) is an \( M \)-alternating path, and \( \overrightarrow{Q} := x_{2p+1}^+ \overrightarrow{Q} x_{2p+1}^+ \) is an \( M' \)-alternating path such that \( |Q'| \equiv 0 \pmod{2} \). Then it follows from Claim 3 that \( A_1 \cup N_C(x_2)^+ \) is independent. Therefore, \( A_1 \cup A_2 \) is an independent set of \( G \).

Finally, we consider the order of \( A_1 \) and \( A_2 \). Suppose that there exist two vertices \( x, y \in N_C(x_1)^+ \cap N_C(z) \) for some \( z \in J'_2 \). Let \( C' := x \overrightarrow{C} y \overrightarrow{x} x \overrightarrow{C} y \overrightarrow{x} x \) and \( M' := M \Delta E(P) \). Then \( |C'| = |C| + 2 \) and \( |M'| = |M| \), contradicting the choice \((C1)\). Therefore, \( |N_C(z) \cap N_C(x_1)^+| \leq 1 \) for any \( z \in J'_2 \). Hence

\[ |I_1| \geq d_C(x_1) - \sum_{z \in J'_2} |N_C(z) \cap N_C(x_1)^+| \geq d_C(x_1) - |J'_2|. \] (3.5)
It follows from the equalities (3.3), (3.4) and (3.5) that

\[
|A_1| = |K_1| + |I_1| + |N_P(x_1)|
\]

\[
\geq |K_1| + d_C(x_1) - |J'_2| + d_P(x_1)
\]

\[
= |K_1| + d_C(x_1) - (|K_1| + 1) + d_H(x_1)
\]

\[
= d_G(x_1) - 1 \geq \delta(G) - 1,
\]

\[
|A_2| = |J'_2| + |N_C(x_2)^+| + |\{v^+, v^{+3}, \ldots , v^{+(2p-1)}\}|
\]

\[
= |K_1| + 1 + d_C(x_2) + p
\]

\[
= \left( |N_H(x_2)| - |V(P) \cap X_1| \right) + 1 + d_C(x_2) + |V(P) \cap X_1| - 1
\]

\[
= d_G(x_2) \geq \delta(G).
\]

This completes the proof of Theorem 1.3. □

4 Conclusion

In this paper, we solve the problem posed by Favaron, Mago and Ordaz [6]. Though several researchers have studied the (almost) balanced independence number condition for a balanced bipartite graph to be Hamiltonian, no one succeeded to get the sharp upper bound on it. In fact, the known upper bounds of $\alpha_{BIP}(G)$ (or $\alpha^*_{BIP}(G)$) which imply the Hamiltonicity are only at most $\delta(G)$, whose coefficient is half of the true value. See [1, 6, 7, 11]. (Note that though the upper bound in [11] does not use the minimum degree, we can easily see that it is bounded by $\delta(G)$ as mentioned in Remark 1.6 in Section 1.) The cause not to obtain the sharp upper bound arises from the method they used, that is very “usual” when we consider the Hamiltonicity. In all the above papers, they considered a longest cycle and showed that if it is not a Hamiltonian cycle, then we can find a large (almost) balanced independent set. However, if we consider only a longest cycle, it is difficult to deal with the case where the outside of it has no edges. (This corresponds to Subsection 3.1 in our proof.) Therefore, defining the function $g(\cdot)$ in this paper, we consider not only the length of cycles but also the order of a maximum matching outside the cycle. By this idea, we get a breakthrough, and succeed to give a sharp upper bound on $\alpha_{BIP}(G)$ and $\alpha^*_{BIP}(G)$ to imply the Hamiltonicity.

References


