Disjoint Even Cycles Packing

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Abstract

We generalize the well-known theorem of Corrádi and Hajnal which says that if a given graph $G$ has at least $3k$ vertices and the minimum degree of $G$ is at least $2k$, then $G$ contains $k$ vertex-disjoint cycles. Our main result is the following: for any integer $k$, there is an absolute constant $c_k$ satisfying the following: let $G$ be a graph with at least $c_k$ vertices such that the minimum degree of $G$ is at least $2k$. Then either (i) $G$ contains $k$ vertex-disjoint even cycles, or (ii) $(2k-1)K_1+pK_2 \leq G \leq K_{2k-1}+pK_2$ ($p \geq k \geq 2$), or $k=1$ and each block in $G$ is either a $K_2$ or an odd cycle, especially, each endblock in $G$ is an odd cycle. In fact, our proof implies the following: the “even cycles” in the conclusion (i) can be replaced by “theta graphs”, where a theta graph is a graph that has two vertices $x$, $y$ such that there are three disjoint paths between $x$ and $y$. Let us observe that if there is a theta graph, then there is an even cycle in it. Furthermore, if the conclusion (ii) holds, clearly there are no $k$ vertex-disjoint even cycles (and hence no $k$ vertex-disjoint theta graphs).

Key Words: Even cycle, Theta graph, Vertex-disjoint

1 Introduction

Packing and covering vertex-disjoint cycles is one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [5] in early 1960s.

**Theorem 1** For any $k$ and any graph $G$, either $G$ has $k$ vertex-disjoint cycles or a vertex set $X$ of order at most $f(k)$ (for some function $f$ of $k$) such that $G - X$ is a forest.

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In fact, Theorem 1 gives rise to the well-known Erdős-Pósa property. A family $\mathcal{F}$ of graphs is said to have the Erdős-Pósa property, if for every integer $k$ there is an integer $f(k, \mathcal{F})$ such that every graph $G$ contains either $k$ vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set $C$ of at most $f(k, \mathcal{F})$ vertices such that $G - C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term Erdős-Pósa property arose because of Theorem 1 which proves that the family of cycles has this property.

Theorem 1 concerns about both “packing”, i.e, $k$ vertex-disjoint cycles and “covering”, i.e, at most $f(k)$ vertices that hit all the cycles in $G$. Starting with this result, there are a host of the results in this direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept “feedback set” in theoretical computer science. Also, the cycle packing problem, which asks whether or not there are $k$ vertex-disjoint cycles in an input graph $G$, is a well-known problem too, e.g. [9].

In graph theory, there are many results concerning packing cycles. The following is the well-known theorem due to Corrādi and Hajnal [3] in 1960s.

**Theorem 2** Let $k$ be an integer, and let $G$ be a graph with at least $3k$ vertices. If the minimum degree of $G$ is at least $2k$, then $G$ contains $k$ vertex-disjoint cycles.

Theorem 2 tells us that if we assumed that the minimum degree of a given graph is at least $2k$, then the covering result in Theorem 1 would not happen. Starting with this result, an enormous number of results, saying “the minimum degree condition” guarantees the existence of packing, appear in the literature.

Our motivation is to consider the parity version of Theorems 1 and 2. Let us first consider the odd cycle case. As observed by Lovász and Schrijver (see [10]), there is a graph that does not have two vertex-disjoint odd cycles but needs at least $\sqrt{n}$ vertices to cover all odd cycles. Hence the Erdős-Pósa property does not hold for odd cycles in general. Also, it is easy to see that in order to guarantee the existence of $k$ vertex-disjoint odd cycles, the minimum degree at least $(n + k)/2$ is needed (just consider a graph obtained from the complete bipartite graph $K_{(n-k+1)/2, (n-k+1)/2}$ by adding $k - 1$ vertices, and join all the edges between these $k - 1$ vertices and the complete bipartite graph). Then this graph does not contain $k$ vertex-disjoint odd cycles, yet the minimum degree is $n/2$. Since the minimum degree in Theorem 2 needs only $2k$ (which does not depend on the order of a graph $G$), thus this makes a huge difference.

On the other hand, the situation is quite different if we consider the even cycle case. Thomassen [11] proved that the Erdős-Pósa property holds for even cycles. In view of Theorems 1 and 2, one might expect that a result like Theorem 2 for even cycles would hold. In fact, we have conjectured that the minimum degree condition in Theorem 2 is, more or less, enough. Our prediction is almost true. The following is our main result.

**Theorem 3** For any integer $k$, there is an absolute constant $c_k$ satisfying the following: let $G$ be a graph with at least $c_k$ vertices such that the minimum degree of $G$ is at least $2k$. Then either

1. $G$ contains $k$ vertex-disjoint even cycles, or
2. $(2k - 1)K_1 + pK_2 \subseteq G \subseteq K_{2k-1} + pK_2 \ (p \geq k \geq 2)$, or $k = 1$ and each block in $G$ is either a $K_2$ or a odd cycle, especially, each endblock in $G$ is a odd cycle.
Since the minimum degree in Theorem 2 is best possible, so the minimum degree in Theorem 3 is also best possible. Furthermore, if the conclusion (ii) holds, clearly there are no \( k \) vertex-disjoint even cycles. So if the minimum degree of a graph \( G \) is at least \( 2k + 1 \), and \( G \) has at least \( c_k \) vertices, then \( G \) always contains \( k \) vertex-disjoint even cycles.

But the order condition \( c_k \) would not be best possible. Let us observe that \( n \geq 4k \) is definitely needed (as the smallest even cycle is a \( C_4 \)), and when \( n = 4k \), finding \( k \) vertex-disjoint even cycles is exactly finding \( k \) vertex-disjoint \( C_4 \). This was conjectured by Erdős and Faudree [4], and has been open for more than 20 years. Recently, Wang announced a solution for this conjecture [12]. We do not know what the best possible \( c_k \) is.

### 2 Technical results

When we prove Theorem 3, we actually consider a stronger statement. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. We are interested in finding \( k \) vertex-disjoint theta graphs, because a theta graph always contain an even cycle, and hence \( k \) vertex-disjoint theta graphs give rise to \( k \) vertex-disjoint even cycles.

The following result is our technical result, which implies Theorem 3.

**Theorem 4** Let \( k \) be an integer, and let \( G \) be a graph with at least \( c_k \) vertices such that the minimum degree of \( G \) is at least \( 2k + 1 \). Then either

1. \( G \) contains \( k \) vertex-disjoint theta graphs, or
2. \((2k - 1)K_1 + pK_2 \subseteq G \subseteq K_{2k-1} + pK_2 \) (\( p \geq k \geq 2 \)), or \( k = 1 \) and each block in \( G \) is either a \( K_2 \) or a cycle, especially, each endblock in \( G \) is a cycle,

where \( c_k := 4(3k + 2)a^{3k-1}(3(k - 1))^{2(k-1)} + (12 + a)(k - 1) \) (\( a = 9600k^2 + 36k + 11 \)).

Let us observe that the smallest theta graph is \( K_4^{-} \), which is obtained from \( K_4 \) by deleting one edge. When \( n = 4k \), it is shown in [7] that if the minimum degree of \( G \) is at least \( 5k/2 \), then \( G \) always contains \( k \) vertex-disjoint \( K_4^{-} \), and the minimum degree condition is best possible. This makes difference between Theorems 3 and 4 when \( n = 4k \) because in Theorem 3, the minimum degree \( 2k \) is sufficient. In general, when \( n \geq 4k \), if the minimum degree of a graph \( G \) is at least \( 2k + 1 \), then there are \( k \) vertex-disjoint \( K_4^{-} \), and the minimum degree condition is best possible [8]. But on the other hand, our result tells us that if we just relax to a theta graph, then the minimum degree \( 2k + 1 \) (which only depends on \( k \)) is enough, provided that a given graph has at least \( c_k \) vertices.

At the moment, we do not know what the best possible bound \( c_k \) would be.

There are some related results in this direction. A chorded cycle is a cycle with at least one chord. A chorded cycle is a theta graph, but the converse is not true. In [1, 6], it is shown that if the minimum degree of a given graph \( G \) (with at least \( 4k \) vertices) is at least \( 3k \), then there are \( k \) vertex-disjoint chorded cycles, and the minimum degree condition is best possible. Thus our result may be considered that if we relax a chorded cycle to a theta graph, then the minimum degree condition can be relaxed too, and it almost matches that for \( k \) vertex-disjoint cycles as in Theorem 2.
3 Sketch of Proof

In this section, we give a very high level sketch of the proof. For more details, we refer the reader to our full version [2].

By way of a contraction, suppose Theorem 4 is not true. Let $G$ be an edge-maximal counterexample, i.e., adding any edge results in satisfying Theorem 4. Thus $G$ contains $k-1$ vertex-disjoint theta graphs, say $\theta_1, \theta_2, \ldots, \theta_{k-1}$. We may assume that $|V(\theta_1)| \leq |V(\theta_2)| \leq \cdots \leq |V(\theta_{k-1})|$. Let $\Theta := \bigcup_{i=1}^{k-1} \theta_i$ and $H := G - \Theta$. We choose $\theta_1, \theta_2, \ldots, \theta_{k-1}$ so that $|V(\Theta)|$ is as small as possible. Since $H$ does not contain a theta graph, it follows that $H$ is disjoint unions of so-called a “block tree”. In other words, each block in $H$ is either a $K_2$ or a cycle. Hence, if $k = 1$, then we can easily see that statement (ii) holds. Thus $k \geq 2$. Moreover, it can be shown that $|V(H)| \geq |V(H)|/4$ (here let $V(F) := \{v \in V(F) : d_F(v) \leq k\}$ for a graph $F$). Otherwise, there is a theta graph in $H$, which gives rise to $k$ vertex-disjoint theta graphs in $G$ (and hence a contradiction). We need to divide the proof into two cases.

Case 1: $|V(\theta_{k-1})| \geq 9600k^2 + 36k + 12$.

Let $H' := G[V(H)] \cup V(\theta_{k-1})$ (here $G[S]$ denote the subgraph of $G$ induced by $S \subseteq V(G)$). For each component $C$ of $H$, let $T_C := \{v \in V(C) : N_{G}(v) \cap V(\theta_{k-1}) \neq \emptyset\}$. Then the following claim holds.

Claim 3.1 Let $C$ be component of $H$. If $|T_C| \geq 1$, then $|V(C) \cap V(H')| \geq 1$, moreover, if $|T_C| \geq 3$, then $|V(C) \cap V(H')| \geq |V(\theta_{k-1})|/12 - 7$.

By Claim 3.1 and the assumption of Case 1, it follows that $|V(H')| \geq 1$ is sufficient large compared to $k$. Thus one of the following happens:

1. There exist a vertex set $W \subseteq V(H') \leq 2$, and two theta graphs $\theta_i, \theta_j$ with $1 \leq i < j \leq k-2$, such that $|W|$ is sufficient large compared to $k$, and $|E_G(\{w\}, V(\theta_i))| \geq 3$, $|E_G(\{w\}, V(\theta_j))| \geq 3$ for any $w \in W$, or

2. there exist a vertex set $W \subseteq V(H') \leq 2$ and a theta graph $\theta_i$ with $1 \leq i \leq k-2$ such that $|W|$ is sufficient large compared to $k$ and $|E_G(\{w\}, V(\theta_i))| \geq 4$ for any $w \in W$.

(For disjoint subsets $S$ and $T$ of $V(G)$, we let $E_G(S, T)$ denote the set of edges of $G$ joining a vertex in $S$ and vertex in $T$.)

In both cases, we can modify $W \cup V(\theta_i) \cup V(\theta_j) \cup V(\theta_k)$ for the second case so that $W \cup V(\theta_i) \cup V(\theta_j)$ contains three vertex-disjoint theta graphs in the first case, and $W \cup V(\theta_k)$ contains two vertex-disjoint theta graphs in the second case, respectively, and hence $G$ contains $k$ vertex-disjoint theta graphs, a contradiction.

Case 2: $|V(\theta_{k-1})| \leq 9600k^2 + 36k + 11$.

By the assumption of Case 2, it follows that $|V(H)|$ is sufficiently large compared to $k$, and consequently, $|V(H)| \leq 2$ is large compared to $k$ (recall that $|V(H)| \leq |V(H)|/4$). Since $|V(\Theta)| \leq (k-1)|V(\theta_{k-1})|$, it follows from the Pigeonhole Principle that there exist $U \subseteq V(\Theta)$ and $W \subseteq V(H)$ such that $|U| = 2k - 2$, $|W| \geq 3k + 2$ and $U \subseteq N_{G}(w)$ for any $w \in W$. Let $G' := G[V(G) - U]$ and $W^* := \{w \in V(G') : U \subseteq N_{G}(w)\}$. Since $W \subseteq W^*$, $|W^*| \geq 3k + 2$. Then the following claim holds.

Claim 3.2 $G'$ contains no two vertex-disjoint connected subgraphs $F_1$ and $F_2$ such that $|V(F_1) \cap W^*| \leq 3$ and $G[V(F_i) \cup \{u\}]$ contains a theta graph for any $u \in U$ (1 \leq i \leq 2).
Proof. Let \( u_1, u_2 \in U \) with \( u_1 \neq u_2 \), and let \( U' := U - \{u_1, u_2\} \). If there exist such graphs \( F_1 \) and \( F_2 \), then \( G[V(F_i) \cup \{u_i\}] \) contains a theta graph for \( i = 1, 2 \). Since \( |W^*| \geq 3k + 2 \) and \( |(V(F_1) \cup V(F_2)) \cap W^*| \leq 6 \), \( G[(W^* - (V(F_1) \cup V(F_2))) \cup U'] \) contains \( k - 2 \) vertex-disjoint theta graphs (actually there are \( k - 2 \) disjoint \( K_{2,3} \) such that each contains exactly three vertices of \( W^* \) and two vertices of \( U' \), respectively). Obviously, these theta graphs are vertex-disjoint, a contradiction.

By Claim 3.2, the following claim can be shown.

Claim 3.3 \( G' \) is connected and \( G' \) contains exactly one vertex \( v \) such that \( d_{G'}(v) \geq 3 \). \( \square \)

Since \( |U| = 2k - 2 \), so every vertex in \( G' \) has minimum degree at least two. On the other hand, by Claim 3.3, \( G' \) is connected, and there is exactly one vertex \( v \) that has degree at least three in \( G' \). It follows that \( G' \) is the union of cycles \( C_1, C_2, \ldots, C_p \) such that \( V(C_i) \cap V(C_j) = \{v\} \) for \( 1 \leq i < j \leq p \). Note that \( W^* = \bigcup_{i=1}^{p} V(C_i) - \{v\} \). If there is a cycle \( C_i \) of length at least four, then by analyzing \( G' \), we can find two vertex-disjoint subgraphs that satisfy Claim 3.2. Similarly, we can show that \( \Theta - U \) consists of induced independent edges. Thus \( |V(C_i)| = 3 \) for \( 1 \leq i \leq p \), and hence \( (2k - 1)K_1 + pK_2 \subseteq G \subseteq K_{2k-1} + pK_2 \), where \( (2k - 1)K_1 \subseteq G\{v\} \cup U \subseteq K_{2k-1} \).

\( \square \)

References


