Minimal Legendrian surfaces in the 5-dimensional Heisenberg group and Weierstrass representation

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We consider *Legendrian surfaces* in the 5-dimensional Heisenberg group $\mathfrak{H}^5$. We obtain a *representation formula* for Legendrian surfaces in $\mathfrak{H}^5$, in terms of spinors [AA]. For especially *minimal* Legendrian surfaces, such data are holomorphic. In this note, we give only the formula for minimal Legendrian surfaces in $\mathfrak{H}^5$. We can regard this formula as an analogue (in Contact Riemannian Geometry) of *Weierstrass representation* for minimal surfaces in $\mathbb{R}^3$. Hence for minimal ones in $\mathfrak{H}^5$, there are many similar results to those for minimal surfaces in $\mathbb{R}^3$. In particular, we give a *Halfspace Theorem* for *properly immersed* minimal Legendrian surfaces in $\mathfrak{H}^5$.

The theory of minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$ is very rich, deep and beautiful. One of the reasons is that every minimal surface in $\mathbb{R}^3$ is represented in terms of holomorphic data, that is, *Weierstrass representation.* Kenmotsu generalized this representation to that for surfaces with prescribed mean curvature. But in general, the data appearing there are not holomorphic anymore, including the case of surfaces with nonzero constant mean curvature. Minimal surfaces are very special from this viewpoint. In Symplectic Riemannian Geometry, there is a similar representation to the above. Chen-Morvan proved that there exists an explicit correspondence in the complex 2-space $\mathbb{C}^2$ between minimal Lagrangian surfaces and holomorphic curves (with nondegenerate condition). Indeed, this correspondence is given by exchanging the orthogonal complex structure in $\mathbb{C}^2$ to another one on $\mathbb{R}^4 = \mathbb{C}^2$. It can be regarded as a primitive form of Weierstrass representation for minimal Lagrangian surfaces in $\mathbb{C}^2$. More generally, Hélein-Romon and the first author proved independently that every Lagrangian surface S in $\mathbb{C}^2$ is represented in terms of a plus spinor (or a minus spinor) of the spin$^{\mathbb{C}}$ bundle $(S\times\mathbb{C}^2)\oplus(K_S^{-1}\oplus K_S)$ satisfying the Dirac equation with potential. Here, K_S denotes the canonical complex line bundle of S. Notice that the representation in terms of plus spinors in $\Gamma(S\times\mathbb{C}^2)$ given by the first author is a natural generalization of the one given by Chen-Morvan.

Let $\mathfrak{H}^5$ be the 5-dimensional *Heisenberg group*, that is, $\mathfrak{H}^5$ is $\mathbb{R}^5 = \mathbb{C}^2 \times \mathbb{R} = \{(\mathbf{z} = (z^1, z^2), t) \mid z^i = x^i + \sqrt{-1}y^i \in \mathbb{C}, t \in \mathbb{R}\}$ as a C^∞ manifold, and it has the group structure as $(\mathbf{z}, t)\cdot(\mathbf{z}', t') = (\mathbf{z}+\mathbf{z}', t+t'+2\,\mathrm{Im}(\mathbf{z}\cdot\overline{\mathbf{z}'}))$. Here, $\mathbf{z}_1\cdot\mathbf{z}_2$ denotes the standard $\mathbb{C}$-linear quadratic form. It has also a natural right invariant *contact 1-form* η defined by $\eta = dt - \frac{1}{2}(\mathbf{x}\cdot d\mathbf{y} - \mathbf{y}\cdot d\mathbf{x})$, that is, $\eta\wedge(d\eta)^2 \neq 0$ everywhere on $\mathfrak{H}^5$ and $(R_p)^*\eta = \eta$ for $p = (\mathbf{z}, t) \in \mathfrak{H}^5$. The *contact structure* $H := \mathrm{Ker}\,\eta$ is a codimension one totally non-integrable subbundle of the tangent bundle $T\mathfrak{H}^5$, which is spanned by the basis $\{T_i := \partial_{x^i} - \frac{1}{2}y^i\partial_t, T_{2+i} := \partial_{y^i} + \frac{1}{2}x^i\partial_t \mid i = 1, 2\}$. Associated with η, there exists a unique vector field $\xi \in \mathfrak{X}(\mathfrak{H}^5)$ with $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$, the so-called *Reeb vector field.* In this case, $\xi = \partial_t$. Then, $T\mathfrak{H}^5$ has a natural decomposition

$$T\mathfrak{H}^5 = H \oplus \mathbb{R}\xi.$$

Let J be the almost complex structure on H defined by $J(T_i) = T_{2+i}, J(T_{2+i}) = -T_i$. We then get an inner product $g_H = d\eta \circ (J \otimes \mathbf{1})$ on H. Since the triple (η, g_H, J) induces the natural Riemannian metric $g = g_\eta$ on $\mathfrak{H}^5$ as

$$g_\eta = \pi_H^* g_H + \eta^2,$$

the triple is called a *contact Riemannian structure* on $\mathfrak{H}^5$. g_η is also called the standard *Sasakian metric* (or *Webster metric*) on $\mathfrak{H}^5$. Here, $\pi_H : T\mathfrak{H}^5 \to H$ is the natural projection associated with the above decomposition. A surface M of $\mathfrak{H}^5$ is said to be *Legendrian* if $T_pM \subset H_p$ for any $p \in M$, which is an integral submanifold of the distribution H with maximal dimension.

With these understandings, the below is the representation formula for minimal Legendrian surfaces in $(\mathfrak{H}^5, g_\eta)$.

Weierstrass Representation. *Let M be a (simply connected) Riemann surface with an isothermal coordinate $w = u + \sqrt{-1}v$ around each point. Let $F = (F_1, F_2) : M \to \mathbb{C}^2$ be a holomorphic map satisfying $|S_1|^2 + |S_2|^2 \neq 0$ everywhere on M, where $S_1 := (F_2)_w$ and $S_2 := -(F_1)_w$. For a constant β, set $f : M \to \mathbb{C}^2 \times \mathbb{R}$ as*

$$\begin{cases} f = \left(\frac{1}{\sqrt{2}} e^{\sqrt{-1}\beta/2}(F_1 - \sqrt{-1}\ \overline{F_2}), \frac{1}{\sqrt{2}} e^{\sqrt{-1}\beta/2}(F_2 + \sqrt{-1}\ \overline{F_1}), t(w)\right), \\ t(w) = -\frac{1}{2} \operatorname{Re} \int^w (F_1 S_1 + F_2 S_2) dw . \end{cases}$$

Then, f is a minimal Legendrian conformal immersion from M to $(\mathfrak{H}^5, g_\eta)$. The induced metric ds^2 on M by f and its Gauss curvature K are respectively given by

$$ds^2 = (|S_1|^2 + |S_2|^2)|dw|^2, \qquad K = -2 \frac{|S_1(S_2)_w - S_2(S_1)_w|^2}{(|S_1|^2 + |S_2|^2)^3}.$$

Conversely, every minimal Legendrian immersion $f : M \to (\mathfrak{H}^5, g_\eta)$ is congruent with the one constructed as above.

Remark. For a minimal Legendrian conformal immersion $f : M \to (\mathfrak{H}^5, g_\eta)$, $\pi_{\mathbb{C}^2} \circ f : M \to \mathbb{C}^2$ is a minimal Lagrangian immersion, where $\pi_{\mathbb{C}^2} : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$ denotes the natural projection. In the above representation formula, the constant β is the Lagrangian angle of $\pi_{\mathbb{C}^2} \circ f$. The Gauss map g of $\pi_{\mathbb{C}^2} \circ f$ is given by

$$g = [-S_2; S_1] = (-S_2/S_1) : M \to \mathbb{C}P^1 = \widehat{\mathbb{C}} \cong S^2.$$

Here, by the identification of S^2 with $S^2(1) \times \{(e^{\sqrt{-1}\beta/2}, 0)\} \subset \mathbb{R}^3 \times \mathbb{R}^3$, g can be regarded as the generalized Gauss map for the Lagrangian surface $(\pi_{\mathbb{C}^2} \circ f)(M)$ in $\mathbb{R}^4 = \mathbb{C}^2$. Set a holomorphic 1-form $hdw := S_1 dw$ on M. In terms of the Gauss data (hdw, g) of M, the induced metric ds^2 and the Gauss curvature K can be rewritten respectively by

$$ds^2 = |h|^2(1 + |g|^2)|dw|^2, \qquad K = -2\Big(\frac{|g_w|}{|h|(1 + |g|^2)^{3/2}}\Big)^2.$$

(See [AAK] for the precise maximal number of exceptional number the Gauss map of a complete minimal Lagrangian surface in $\mathbb{C}^2$.)

Set $\mathfrak{H}^5_{\geq t_0} = \{p \in \mathfrak{H}^5 \mid t(p) \geq t_0\}$ and $\mathfrak{H}^5_{\leq t_0} = \{p \in \mathfrak{H}^5 \mid t(p) \leq t_0\}$ for $t_0 \in \mathbb{R}$. The following is a Legendrian version of the Halfspace Theorem for minimal surfaces in $\mathbb{R}^3$.

Halfspace Theorem. *Any properly immersed minimal Legendrian surface in* $(\mathfrak{H}^5, g_\eta)$ *contained in an upper half space* $\mathfrak{H}^5_{\geq t_0}$ *(or a lower half space* $\mathfrak{H}^5_{\leq t_0}$*) is a Legendrian plane contained in* $\{p \in \mathfrak{H}^5 \mid t(p) = t_1\} = \mathbb{C}^2 \times \{t_1\}$ *for some* $t_1 \in \mathbb{R}$*. Moreover, it is also a Lagrangian plane in* $\mathbb{C}^2 = \mathbb{C}^2 \times \{t_1\}$*.*

References

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