On Fluid Flow on a Moving Hypersurface
(動く曲面上における流体の流れに関して)*

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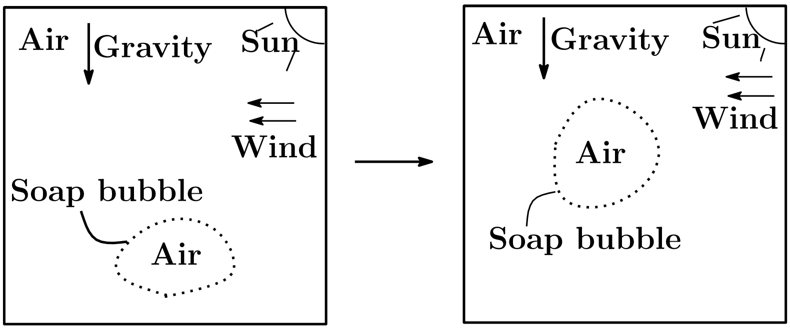
Abstract. This paper studies various incompressible fluid systems from an energetic point of view. Here incompressibility means that the density of the fluid is a constant. In particular, we consider fluid flow on a moving hypersurface. A moving hypersurface means that the hypersurface is moving or the shape of the hypersurface is changing along with the time. From an energetic point of view, we study the dominant equations for the motion of the fluid on a moving hypersurface.

1. Introduction

Image a soap bubble in air in the sun:



Air: 空気, Soap bubble: シャボン玉, Gravity: 重力,
Sun: 太陽, Wind: 風

Figure 1. Soap Bubble in Air in the Sun

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Of course, a soap bubble is floated by the wind and the shape of the bubble is changed by the gravity. When we focus on the soap bubble, we can see fluid flow in the bubble. The flow is often called a *surface flow* or an *interfacial flow*. Surface flow and surface tension play an important role in a soap bubble in air. One can consider surface flow as fluid flow on a moving hypersurface. We are interested in the dominant equations for the motion of the fluid on a moving hypersurface.

In this paper we study various fluid systems from an energetic point of view. More precisely, we consider fluid systems in a domain, on a manifold, and on a moving hypersurface. In particular, we treat the case when the density of the fluid is a constant. This property is often called *incompressibility*. In this paper we introduce incompressible fluid system on a moving hypersurface, derived by our energetic variational approach.

Outline of this paper is as follows: In Section 2 we study basic terminology in fluid dynamics, and consider both kinetic and dissipation energies of incompressible fluid systems in a domain. In Section 3 we introduce the incompressible inviscid fluid system, derived by Arnold [2, 3], on a manifold and the incompressible viscous fluid system, derived by Taylor [7], on a manifold. Section 3 describes Taylor's argument for deriving his incompressible fluid system. In Section 4 we introduce incompressible fluid systems on a moving hyperspace, which is derived by our energetic variational approach. We state main results and key ideas for incompressible inviscid fluid system on a moving hypersurface. In the final section, we refer to the comparison between our models and the previous models.

2. Fluid Mechanics

Let us first study basic terminology in fluid dynamics. When we study fluid flow in a domain or on a surface, we need to consider the density of the fluid, the velocity of the fluid, the pressure of the fluid, and the viscosity of the fluid. In fluid dynamics, fluid is classified as follows:

$$\text{Fluid} - \begin{cases} \text{Inviscid fluid [Ideal fluid]} \\ \text{Viscosity fluid[Liquid(Water), Gas(Air)]} \end{cases}$$

$$\text{Fluid} - \begin{cases} \text{Compressible fluid [Density is a function]} \\ \text{Incompressible fluid [Density is a constant]} \end{cases}$$

Remark: To be exact, we call fluid *incompressible fluid* if the material derivative of the density equals to zero, that is, $D_t\rho = 0$. Here D_t denotes the material derivative (see this latter part).

From now we consider incompressible fluid flow in a domain, on a manifold, and on a moving hypersurface. In other words, we treat the case when the density of the fluid is a constant. In particular, we study the incompressible Euler and Navier-Stokes systems:

$$\text{Fluid systems} - \begin{cases} \text{Euler system [Inviscid fluid system]} \\ \text{Navier-Stokes system [Viscosity fluid system]} \end{cases}$$

流体力学の用語

Fluid flow: 流体の流れ，Velocity $v = {}^t(v_1, v_2, v_3)$: 流体の速さ
Pressure p: 流体の圧力，Density ρ: 流体の密度
Viscosity μ: 流体の粘性 (係数)
Inviscid fluid: 非粘性流体，Viscosity fluid: 粘性流体
Compressible fluid: 圧縮性流体, Incompressible fluid:非圧縮性流体
Euler system: 非粘性流体の流れを支配する流体方程式
Navier-Stokes system: 粘性流体の流れを支配する流体方程式
Incompressible Euler system: 非粘性非圧縮性流体方程式
Incompressible Navier-Stokes system: 粘性非圧縮性流体方程式

Let us introduce the well-known incompressible Euler system and incompressible Navier-Stokes system in a domain. Let Ω be a domain in $\mathbb{R}^3$. Let $t \geq 0$ be the time variable and $x = {}^t(x_1, x_2, x_3) \in \mathbb{R}^3$ be the spatial variables.

The symbols $v = v(x,t) = {}^t(v_1(x,t), v_2(x,t), v_3(x,t))$, $p = p(x,t)$, and $\rho = \rho(x,t)$ represent the velocity of the fluid in the domain Ω, the pressure of the fluid in Ω, and the density of the fluid in Ω, respectively. The symbol μ is the viscosity coefficient of the fluid in Ω. Assume that v, p are smooth functions and that ρ_0, μ are two positive constants. Suppose that $\rho \equiv \rho_0$.

Incompressible Euler system in a domain

$$\text{(E)} \begin{cases} \rho_0 D_t v + \text{grad} p = 0, \\ \text{div} v = 0. \end{cases}$$

Incompressible Navier-Stokes system in a domain

$$\text{(NS)} \begin{cases} \rho_0 D_t v + \text{grad} p = \mu \Delta v, \\ \text{div} v = 0. \end{cases}$$

Here

$$\begin{cases} \partial_t = \frac{\partial}{\partial t} & : \text{ time derivative,} \\ \partial_i = \frac{\partial}{\partial x_i} & : \text{ space derivative,} \\ \nabla = {}^t(\partial_1, \partial_2, \partial_3) & : \text{ space derivative,} \\ \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2 & : \text{ Laplacian,} \\ (v, \nabla) f = v_1 \partial_1 f + v_2 \partial_2 f + v_3 \partial_3 f & : \text{ derivative along with velocity } v, \\ D_t f = \partial_t f + (v, \nabla) f & : \text{ material derivative,} \\ \mathrm{grad} p = \nabla p = {}^t(\partial_1 p, \partial_2 p, \partial_3 p) & : \text{ gradient,} \\ \mathrm{div} v = \nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 & : \text{ divergence.} \end{cases}$$

We call $D_t v$ the *nonlinear part*, grad$p$ the *pressure part*, and $\mu \Delta v$ the *viscous part* of the fluid system.

Let us consider the viscous part $\mu \Delta v$ and some energies of the fluid in a domain. Now we assume that ρ is a function. Write

$$D(v) = \frac{1}{2}\{(\nabla v)^T + \nabla v\}.$$

We call $D(\cdot)$ a *deformation tensor*. A direct calculation shows that

$$2\mu \mathrm{div} D(v) = \mu \Delta v \text{ if } \mathrm{div} v = 0.$$

Set

$$e_K = \frac{1}{2}\rho |v|^2 \text{ and } e_D = 2\mu D(v) : D(v).$$

We call e_K *kinetic energy* and e_D *dissipation energy* of incompressible fluid in a domain.

Let us derive the kinetic and dissipation energies. Assume that $v \in C((0,T); [W_0^{1,2}(\Omega)]^3)$. Multiplying the system (NS) by the v, and then integrating by parts shows that for all $t > s \geq 0$

$$\int_\Omega \frac{1}{2}\rho(x,t)|v(x,t)|^2 dx + \int_s^t \int_\Omega \mu D(v(x,\tau)) : D(v(x,\tau)) dx d\tau$$
$$= \int_\Omega \frac{1}{2}\rho(x,s)|v(x,s)|^2 dx \ \cdots (\mathrm{EE}).$$

We call (EE) an *energy equality.* From (EE) we can derive the following *asymptotic stability*:

$$\lim_{t\to\infty} \int_\Omega |v(x,t)|^2 dx = 0 \ \cdots (\mathrm{AS}).$$

Remark: It is easy to show (AS) if Ω is a bounded domain. It is not easy to derive (AS) if Ω is an unbounded domain.

用語: Deformation tensor: 変形テンソル, Kinetic energy: 運動エネルギー, Dissipation energy: 散逸エネルギー

3. Incompressible Fluid Flow Systems on Manifold

Let us study the incompressible Euler system on a manifold derived by Arnold [2, 3] and the incompressible Navier-Stokes system on a manifold introduced by Taylor [7].

Let $\mathcal{M}$ be a closed Riemannian 2-dimensional manifold, and let $\mu > 0$ be the viscosity coefficients of the fluid on $\mathcal{M}$. Let ρ be the density of the fluid on $\mathcal{M}$. Let u be the velocity of the fluid on $\mathcal{M}$, and let p be a pressure associated with u. Suppose that $\rho \equiv 1$. Assume that u is a 1-form on $\mathcal{M}$ and that p is a function on $\mathcal{M}$.

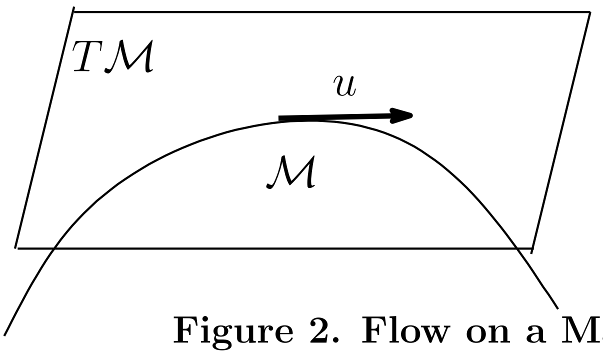

Figure 2. Flow on a Manifold

Arnold [2, 3] applied the kinetic energy $\frac{1}{2}|u|^2$ and the Lie group of diffeomorphisms to derive the following Euler system on a manifold $\mathcal{M}$:

$$(\mathrm{E})_{\mathcal{M}} \begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = 0, \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

See also Ebin-Marsden [4].

Taylor [7] introduced the following Navier-Stokes system, derived from their physical sense, on a manifold $\mathcal{M}$:

$$(\mathrm{NS})_{\mathcal{M}} \begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = \mu(\Delta_{\mathcal{M}} u + K u), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Here

$$\begin{cases} \Delta_{\mathcal{M}} & : \text{ the Bochner-Laplacian}, \\ K & : \text{ the Gaussian curvature (the Ricci curvature)}, \\ \mathrm{grad}_{\mathcal{M}} & : \text{ gradient operator on } \mathcal{M}, \\ \mathrm{div}_{\mathcal{M}} & : \text{ divergence operator on } \mathcal{M}, \\ \nabla_u u & : \text{ covariant derivative along with the velocity } u. \end{cases}$$

Remark: The operators $\Delta_{\mathcal{M}}$, $\mathrm{grad}_{\mathcal{M}}$, and $\mathrm{div}_{\mathcal{M}}$ are defined by exterior derivatives.

Mitsumatsu and Yano [6] also derived the system $(\mathrm{NS})_{\mathcal{M}}$ by using their energetic variational approach. Arnaudon and Cruzeiro [1] applied stochastic variational approach to derive the system $(\mathrm{NS})_{\mathcal{M}}$.

Let us roughly explain Taylor's method for deriving the Navier-Stokes system on a manifold. Set

$$D_{\mathcal{M}}(u) = \frac{1}{2}\{(\nabla_{\mathcal{M}} u)^T + \nabla_{\mathcal{M}} u\}.$$

Here $\nabla_{\mathcal{M}}$ is the covariant derivative operator on $\mathcal{M}$.
If $\mathrm{div}_{\mathcal{M}} u = 0$ and u is a 1-form, then

$$2\mathrm{div}_{\mathcal{M}} D_{\mathcal{M}}(u) = \mathrm{div}_{\mathcal{M}}\{(\nabla_{\mathcal{M}} u)^T + (\nabla_{\mathcal{M}} u)^T\} = \Delta^B v + \mathrm{Ric}(u),$$

where $\Delta^B = -\nabla_{\mathcal{M}}^* \nabla_{\mathcal{M}}$ (Bochner-Laplacian, Laplace-Beltrami) and Ric is the Ricci operator: $T\mathcal{M}$ into itself. Here $\nabla_{\mathcal{M}}^*$ is the adjoint operator with respect to $L^2(\mathcal{M}, T\mathcal{M})$.

Taylor assumed that (u, p) satisfy

$$\begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = 2\mu \mathrm{div}_{\mathcal{M}} D_{\mathcal{M}}(u), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Then

$$\begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = \mu \Delta^B u + \mu \mathrm{Ric}(u), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Using the Weitzenböck formula:

$$\Delta^B u = \Delta^H u + \mathrm{Ric}(u),$$

we have

$$(\mathrm{NS})_{\mathcal{M}} \begin{cases} \partial_t u + \nabla_u u + \mathrm{grad}_{\mathcal{M}} p = \mu \Delta^H u + 2\mu \mathrm{Ric}(u), \\ \mathrm{div}_{\mathcal{M}} u = 0. \end{cases}$$

Here Δ^H is the Hodge Laplacian (Laplace-deRham operator).

Remark: Mitsumatsu-Yano [6] and Arnaudon-Cruzeiro [1] considered

$$2\mu D_{\mathcal{M}}(u) : D_{\mathcal{M}}(u)$$

as the dissipation energy of the incompressible fluid on a manifold to derive the viscous part of the system $(\mathrm{NS})_{\mathcal{M}}$:

$$2\mu \mathrm{div}_{\mathcal{M}} D_{\mathcal{M}}(u).$$

4. Fluid Flow Systems on Moving Hypersurface

In this section we introduce incompressible fluid systems on a moving hypersurface, derived by our energetic variational approach. Let $x = {}^t(x_1, x_2, x_3)$ be the spatial variable in $\mathbb{R}^3$ and $t \geq 0$ be the time variable. Let $\Gamma(t)$ be a hypersurface in $\mathbb{R}^3$ depending on time $t \in [0, T)$ for some $T \in (0, \infty]$. Let $w = {}^t(w_1, w_2, w_3)$ be the motion velocity of $\Gamma(t)$, and $u = {}^t(u_1, u_2, u_3)$ be a relative velocity on $\Gamma(t)$.

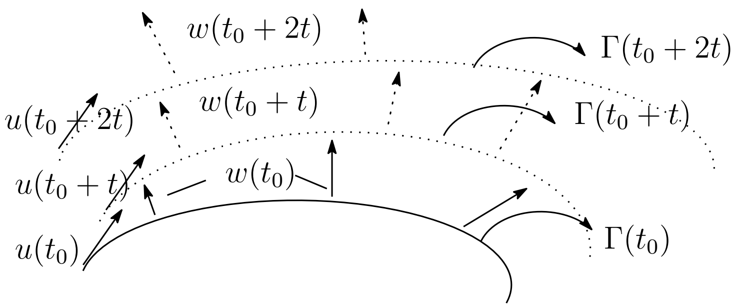

Figure 3. Flows on a Moving Hypersurface

The velocity

$$v = {}^t(v_1, v_2, v_3) := u + w$$

is called a *total velocity* of the fluid on $\Gamma(t)$. We focus on the total velocity v. The symbol q denotes a *total pressure* or a *pressure associated with* v. Let ρ and μ be the density and viscosity of the fluid on $\Gamma(t)$, respectively. Write

$$\mathcal{S}_T = \left\{ (x,t) \in \mathbb{R}^4;\ (x,t) \in \bigcup_{0<t<T} \left\{ \Gamma(t) \times \{t\} \right\} \right\}.$$

We assume that $\Gamma(t)$ is a 2-dimensional closed manifold for each fixed $t \in [0, T)$. Set

$$C^\infty(\mathcal{S}_T) = \{ f : \mathcal{S}_T \to \mathbb{R};\ f = g|_{\mathcal{S}_T} \text{ for some } g \in C^\infty(\mathbb{R}^4) \},$$
$$C_0^\infty(\mathcal{S}_T) = \{ f \in C^\infty(\mathcal{S}_T);\ \ \operatorname{supp} f(\cdot, t) \subset \Gamma(t) \}.$$

Assume that ρ, u, w, v, q are smooth functions represented by

$$\rho = \rho(x,t),\ u = u(x,t),\ w = w(x,t),\ v = v(x,t),\ q = q(x,t),$$

i.e. $\rho, u, w, v, q \in C^\infty(\mathcal{S}_T)$. Suppose that μ is a positive constant.

In order to consider incompressible fluid flow on an moving hypersurface, we assume that ρ is a positive constant ρ_0. Moreover, we assume that $0 \le t < T$

$$\int_{\Gamma(t)} H(x,t) n(x,t) \cdot w(x,t) \ d\mathcal{H}^2_x = 0,$$

where H is the mean curvature, n is the unit outer normal vector, and $d\mathcal{H}^2_x$ is 2-dimensional Hausdorff measure.

Now we introduce incompressible fluid system on the moving hypersurface. Applying our energetic variational approach, we derive the following incompressible fluid systems:

Incompressible inviscid fluid system (I)

$$\begin{cases} \rho_0 D_t v + \mathrm{grad}_\Gamma q + qHn = 0, \\ \mathrm{div}_\Gamma v = 0. \end{cases}$$

Incompressible inviscid fluid system (II)

$$(\mathrm{E})_\Gamma \begin{cases} \rho_0 D^\Gamma_t v + \mathrm{grad}_\Gamma q = 0, \\ \mathrm{div}_\Gamma v = 0, \\ v \cdot n = 0. \end{cases}$$

Incompressible viscous fluid system (I)

$$\begin{cases} \rho_0 D_t v + \mathrm{grad}_\Gamma q + qHn = 2\mu \mathrm{div}_\Gamma D_\Gamma(v), \\ \mathrm{div}_\Gamma v = 0. \end{cases}$$

Incompressible viscous fluid system (II)

$$(\mathrm{NS})_\Gamma \begin{cases} \rho_0 D^\Gamma_t v + \mathrm{grad}_\Gamma q = 2\mu P_\Gamma \mathrm{div}_\Gamma D_\Gamma(v), \\ \mathrm{div}_\Gamma v = 0, \\ v \cdot n = 0. \end{cases}$$

Here $n = n(x,t) = {}^t(n_1, n_2, n_3)$: unit outer normal vector,

$$\begin{cases} \rho_0 > 0 : \ \text{density}, \\ v = {}^t(v_1, v_2, v_3) : \ \text{velocity}, \\ q : \ \text{pressure} \ , \\ \mu : \ \text{viscosity}, \\ H : \ \text{mean curvature}, \end{cases} \qquad \begin{cases} \mathrm{div}_\Gamma : \ \text{surface divergence}, \\ \mathrm{grad}_\Gamma : \ \text{surface gradient}, \\ P_\Gamma : \text{orthogonal projection to tangent}, \\ D_t : \ \text{material derivative}, \\ D^\Gamma_t : \ \text{surface material derivative}, \\ D_\Gamma(\cdot) : \ \text{surface deformation tensor}, \end{cases}$$

where

$$\begin{aligned} \partial_i^{tan} &= (\delta_{ij} - n_i n_j)\partial_j, \\ \nabla^{tan} &= {}^t(\partial_1^{tan}, \partial_2^{tan}, \partial_3^{tan}), \\ \mathrm{grad}_\Gamma q &= \nabla^{tan} q = {}^t(\partial_1^{tan} q, \partial_2^{tan} q, \partial_3^{tan} q), \\ \mathrm{div}_\Gamma v &= \nabla^{tan} \cdot v = \partial_1^{tan} v_1 + \partial_2^{tan} v_2 + \partial_3^{tan} v_3, \\ D_t f &= \partial_t f + (v, \nabla) f, \\ D_t^\Gamma f &= \partial_t f + (v, \nabla^{tan}) f. \end{aligned}$$

Remark; Roughly speaking, the two systems $(\mathrm{E})_\Gamma$ and $(\mathrm{NS})_\Gamma$ correspond to fluid systems in the case when the hypersurface is fixed.

Let us state key ideas to derive our incompressible fluid systems. The first one is to focus on both kinetic and dissipation energies of the incompressible fluid on a moving hypersurface. The second one is to apply a flow map on the moving hypersurface. Let us introduce a *flow map* on a moving hypersurface. Let $\Gamma(t)$ be a moving hypersurface. We call a smooth function $x = x(\xi, t) = {}^t(x_1, x_2, x_3)$ a flow map on $\Gamma(t)$ if there is smooth function $v = v(x, t) = {}^t(v_1, v_2, v_3)$ such that for $0 < t < T$ and $\xi \in \Gamma(0)$,

$$\begin{cases} \frac{dx}{dt}(\xi, t) = v(x(\xi, t), t), \\ x(\xi, 0) = \xi. \end{cases}$$

We call v the *velocity* determined by x. We assume that v is the total velocity on $\Gamma(t)$.

Next we state main results for the incompressible inviscid fluid system on a moving hypersurface.

Theorem 1(Incompressible condition)

Let $\Gamma(t)$ be a moving hypersurface and ρ_0 be a positive constant. For each $0 < t < T$ and every $\Omega(t) \subset \Gamma(t)$ is flowed by the velocity vector $v = {}^t(v_1, v_2, v_3)$, assume that

$$\frac{d}{dt} \int_{\Omega(t)} \rho_0 \, d\mathcal{H}_x^2 = 0.$$

Then

$$\mathrm{div}_\Gamma v = 0.$$

Here $d\mathcal{H}_x^2$ denotes the 2-dimensional Hausdorff measure.

Remark: We call the condition that $\mathrm{div}_\Gamma v = 0$ *surface divergence free.* The surface divergence free condition implies surface area preserving of the moving hypersurface $\Gamma(t)$.

Theorem 2(Necessary condition for existence of incompressible flow)

Let $\Gamma(t)$ be a moving hypersurface. Assume that $v = u + w$ and that $u \cdot n = 0$. If

$$\mathrm{div}_\Gamma v = 0,$$

then

$$\int_{\Gamma(t)} H(x,t)n(x,t) \cdot w(x,t) \; d\mathcal{H}^2_x = 0.$$

Here $d\mathcal{H}^2_x$ denotes the 2-dimensional Hausdorff measure.

Remark: The motion condition $\int_{\Gamma(t)} H(x,t)n(x,t) \cdot w(x,t) \; d\mathcal{H}^2_x = 0$ is important for the existence of incompressible flow on the moving hypersurface.

Theorem 3(Variation of kinetic energy)

Let $\Gamma(t)$ be a moving hypersurface and ρ_0 be a positive constant. Let x be a flow map on $\Gamma(t)$ and v be the velocity determined by x. Assume that v is the total velocity on $\Gamma(t)$ and that $\mathrm{div}_\Gamma v = 0$. Let $\widehat{\Gamma}(t)$ be a variation of $\Gamma(t)$. Let $\widehat{x}$ be a flow map on $\widehat{\Gamma}(t)$ and $\widehat{v}$ be the velocity determined by $\widehat{x}$. Suppose that $\widehat{x}$ is variation of x and that $\widehat{v}$ is a variation of v. For each variation $\widehat{x}$ with $\mathrm{div}_\Gamma \widehat{v} = 0$, set

$$Act[\widehat{x}] = \int_0^T \int_{\widehat{\Gamma}(t)} \frac{1}{2}\rho_0 |\widehat{v}(x,t)|^2 \; d\mathcal{H}^2_x dt.$$

A critical point of the action integral $Act[\cdot]$ under area conserving surface (domain) deformation fulfills

$$\rho_0 D_t v + \nabla^{tan} q + qHn = 0$$

for some q. Moreover, assume in addition that $v \cdot n = 0$. Then

$$\rho_0 D_t^\Gamma v + \nabla^{tan} q = 0.$$

Combining Theorems 1-3, we can derive inviscid incompressible fluid systems on a moving hypersurface.

Remark: See Giga-Koba-Liu [5] for the proof of Theorems 1-3, the dissipation energy of incompressible fluid on a moving hypersurface, and the derivation of the viscous parts $2\mu \mathrm{div}_\Gamma D_\Gamma(v)$ and $2\mu P_\Gamma \mathrm{div}_\Gamma D_\Gamma(v)$.

5. Comparison with Previous Models

Let us compare the system $(\mathrm{E})_\Gamma$ with the Euler system $(\mathrm{E})_\mathcal{M}$ derived by Arnold [2, 3] and the system $(\mathrm{NS})_\Gamma$ with the Navier-Stokes system $(\mathrm{NS})_\mathcal{M}$ introduced by Taylor [7]. If $\rho_0 = 1$ and v is a tangent vector on the surface, the incompressible fluid system $(\mathrm{E})_\Gamma$ is noting but the Euler system on a manifold. The system $(\mathrm{E})_\Gamma$ with $\rho_0 = 1$ and $P_\Gamma v = v$ is same as the Euler system on a manifold derived by Arnold. However, when $\rho_0 = 1$, $\mu > 0$, and $P_\Gamma v = v$, our model $(\mathrm{NS})_\Gamma$ is different from the Navier-Stokes system $(\mathrm{NS})_\mathcal{M}$ on a manifold introduced by Taylor. See [5] for details.

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