Remark on Harnack-Type Inequalities for the Porous Medium Equation on Riemannian Manifolds

只野　誉　(阪大・理)*

概　要

In this note, we shall give some local gradient estimates of Li-Yau and Hamilton type for positive solutions to the porous medium equation

$$\frac{\partial u}{\partial t} = \Delta_g u^m, \quad m > 1,$$

on complete Riemannian manifolds (X, g) with Ricci curvature bounded from below. As an application, we shall give some Harnack inequalities for such positive solutions on complete non-compact manifolds. In particular, our results improve recent work by Huang, Huang and Li [3].

1. Introduction

Differential Harnack inequalities for solutions to the heat equation are powerful tools in geometric analysis and originated with the celebrated work [4] by Li and Yau, where they studied positive solutions to the heat equatuon

$$\frac{\partial u}{\partial t} = \Delta_g u \tag{1.1}$$

on complete Riemannian manifold (X, g) with Ricci curvature bounded from below and derived the following gradient estimate for such positive solutions:

Theorem A (Li-Yau [4]). *Let (X, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_g(B_p(2R)) \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x, t)$ is a positive solution to (1.1) on $B_p(2R) \times (0, T]$. Then, on $B_p(R) \times (0, T]$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leqslant \frac{C(n)\alpha^2}{R^2}\left(\frac{\alpha^2}{\alpha - 1} + 1 + \sqrt{K}R\right) + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)} + \frac{n\alpha^2}{2t}, \tag{1.2}$$

where $\alpha > 1$ is any constant and $C(n)$ is a constant depending only on n.

By taking $R \to \infty$ in (1.2) above, we obtain the following global gradient estimate for positive solutions to the heat equation:

Corollary B (Li-Yau [4]). *Let (X, g) be an n-dimensional complete non-compact Riemannian manifold with $\mathrm{Ric}_g \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x, t)$ is a*

部分多様体幾何とリー群作用 2015 (2015 年 9 月 7 日–2015 年 9 月 8 日, 於 東京理科大学) アブストラクト
2010 Mathematics Subject Classification: Primary 35B45, Secondary 35K55
キーワード：熱方程式, Porous Medium 方程式, 勾配評価, Li-Yau Harnack 不等式
* 〒560-0043 大阪府豊中市待兼山町 1 番 1 号 大阪大学大学院理学研究科
e-mail: h-tadano@cr.math.sci.osaka-u.ac.jp

positive solution to (1.1) *on* $X \times (0,T]$. *Then, on* $X \times (0,T]$,

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leqslant \frac{n\alpha^2 K}{\sqrt{2}(\alpha-1)} + \frac{n\alpha^2}{2t}, \tag{1.3}$$

where $\alpha > 1$ *is any constant.*

By integrating the gradient estimate (1.3) along a space-time path, we may obtain the following classical Harnack inequality:

Corollary C (Li-Yau [4])**.** *Let* (X,g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g \geqslant -K$ *for* $K \geqslant 0$. *Suppose that* $u = u(x,t)$ *is a positive solution to* (1.1) *on* $X \times (0,T]$. *Then, for any* $x_1, x_2 \in X$ *and* $0 < t_1 < t_2 < T$,

$$u(x_1,t_1) \leqslant u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n\alpha}{2}} \exp\left\{\frac{n\alpha K}{\sqrt{2}(\alpha-1)}(t_2-t_1) + \frac{\alpha\,\mathrm{dist}^2(x_1,x_2)}{4(t_2-t_1)}\right\},$$

where $\alpha > 1$ *is any constant and* $\mathrm{dist}(x_1,x_2)$ *is the distance between* x_1 *and* x_2.

On the other hand, Hamilton [2] proved the following gradient estimate for positive solutions to the heat equation, which may be compared with Corollary B above:

Theorem D (Hamilton [2])**.** *Let* (X,g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g \geqslant -K$ *for* $K \geqslant 0$. *Suppose that* $u = u(x,t)$ *is a positive solution to* (1.1) *on* $X \times (0,T]$. *Then, on* $X \times (0,T]$,

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{u_t}{u} \leqslant e^{4Kt}\frac{n}{2t}.$$

2. Results

Recently, several generalizations of the heat equation have been studied. In this talk, we shall study the *porous medium equation*

$$\frac{\partial u}{\partial t} = \Delta_g u^m, \quad m > 1, \tag{2.1}$$

which is a non-linear extension of the heat equation (1.1). The porous medium equation (2.1) is of great interest due to their importance in mathematics [1, 7]. For various values of $m > 1$, it arises in different applications to model diffusive phenomena such as groundwater infiltration ($m = 2$), flow of gas in porous media ($m \geqslant 2$), heat radiation in plasmas ($m > 4$), liquid thin films moving under gravity ($m = 4$).

Li-Yau type gradient estimates for positive solutions to the porous medium equation (2.1) were first proved by Lu, Ni, Vázquez and Villani [5].

Theorem E (Lu-Ni-Vázquez-Villani [5])**.** *Let* (X,g) *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}_g(B_p(2R)) \geqslant -K$ *for* $K \geqslant 0$. *Suppose that* $u = u(x,t)$ *is a positive solution to* (2.1) *on* $B_p(2R) \times (0,T]$. *Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1)\sup_{B_p(2R)\times(0,T]} v \quad \textit{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R)\times(0,T]$,

$$\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\leqslant\frac{C(n)a\alpha^2L}{R^2}\left\{\frac{m^2a\alpha^2}{(\alpha-1)(m-1)}+1+\sqrt{K}R\right\}+\frac{a\alpha^2}{t}+\frac{a\alpha^2KL}{\alpha-1},\tag{2.2}$$

where $\alpha>1$ *is any constant and* $C(n)$ *is a constant depending only on* n.

By taking $R\to\infty$ in (2.2), we obtain the following global gradient estimate:

Corollary F (Lu-Ni-Vázquez-Villani [5]). *Let* (X,g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g\geqslant -K$ *for* $K\geqslant 0$. *Suppose that* $u=u(x,t)$ *is a positive solution to* (2.1) *on* $X\times(0,T]$. *Put*

$$v=\frac{m}{m-1}u^{m-1},\quad L=(m-1)\sup_{X\times(0,T]}v\quad\textit{and}\quad a=\frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X\times(0,T]$,

$$\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\leqslant\frac{a\alpha^2}{t}+\frac{a\alpha^2KL}{\alpha-1},$$

where $\alpha>1$ *is any constant.*

On the other hand, Huang, Huang and Li [3] gave the following estimate:

Theorem G (Huang-Huang-Li [3]). *Let* (X,g) *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}_g(B_p(2R))\geqslant -K$ *for* $K\geqslant 0$. *Suppose that* $u=u(x,t)$ *is a positive solution to* (2.1) *on* $B_p(2R)\times(0,T]$. *Put*

$$v=\frac{m}{m-1}u^{m-1},\quad L=(m-1)\sup_{B_p(2R)\times(0,T]}v\quad\textit{and}\quad a=\frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R)\times(0,T]$,

$$\begin{aligned}\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\leqslant a\alpha^2\Bigg\{&\left(\frac{C(n)L}{R^2}\cdot\frac{m^2a\alpha^2}{(\alpha-1)(m-1)}\right)^{\frac12}\\&+\left[\frac{1}{t}+\frac{KL}{2(\alpha-1)}+\frac{C(n)L}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right]^{\frac12}\Bigg\}^2,\end{aligned}\tag{2.3}$$

where $\alpha>1$ *is any constant and* $C(n)$ *is a constant depending only on* n.

By taking $R\to\infty$ in (2.3), we obtain the following global gradient estimate, which improves Corollary F above:

Corollary H (Huang-Huang-Li [3]). *Let* (X,g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g\geqslant -K$ *for* $K\geqslant 0$. *Suppose that* $u=u(x,t)$ *is a positive solution to* (2.1) *on* $X\times(0,T]$. *Put*

$$v=\frac{m}{m-1}u^{m-1},\quad L=(m-1)\sup_{X\times(0,T]}v\quad\textit{and}\quad a=\frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X\times(0,T]$,

$$\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\leqslant\frac{a\alpha^2}{t}+\frac{a\alpha^2KL}{2(\alpha-1)},$$

where $\alpha>1$ *is any constant.*

Moreover, Huang, Huang and Li [3] gave the following estimate of Hamilton type:

Theorem I (Huang-Huang-Li [3]). *Let (X,g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_g(B_p(2R)) \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $B_p(2R) \times (0,T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1)\sup_{B_p(2R)\times(0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0,T]$,

(2.4)

$$\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leqslant \frac{C(n)La\alpha^2(t)}{R^2}\left\{\frac{m^2a\alpha^2(t)}{2(\alpha(t)-1)(m-1)} + 1 + \sqrt{K}R\coth(\sqrt{K}R)\right\} + \frac{a\alpha^2(t)}{t},$$

where $\alpha(t) = e^{2KLt}$ and $C(n)$ is a constant depending only on n.

By taking $R \to \infty$ in (2.4), we obtain the following global gradient estimate, which may be compared with Corollary H above:

Corollary J (Huang-Huang-Li [3]). *Let (X,g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_g \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $X \times (0,T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1)\sup_{X\times(0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0,T]$,

$$\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leqslant \frac{a\alpha^2(t)}{t},$$

where $\alpha(t) = e^{2KLt}$.

In this talk, we further study gradient estimates for positive solutions to the porous medium equation (2.1) on complete Riemannian manifolds with Ricci curvature bounded from below and shall improve Theorem G and I. Now, we state our results as follows:

Theorem 2.5 ([6]). *Let (X,g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_g(B_p(2R)) \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $B_p(2R) \times (0,T]$. Put*

$$v = \frac{m}{m-1}u^{m-1}, \quad L = (m-1)\sup_{B_p(2R)\times(0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0,T]$,

(2.6)

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leqslant a\alpha^2\left\{\left(\frac{C(n)L}{R^2}\cdot\frac{m^2a\alpha^2}{(\alpha-1)(m-1)}\right)^{\frac{1}{2}} + \left[\frac{1}{1+a(\alpha-1)}\cdot\frac{1}{t} + \frac{KL}{2(\alpha-1)} + \frac{C(n)L}{R^2}\left(1+\sqrt{K}R\coth(\sqrt{K}R)\right)\right]^{\frac{1}{2}}\right\}^2,$$

where $\alpha > 1$ is any constant and $C(n)$ is a constant depending only on n.

By taking $R \to \infty$ in (2.6), we obtain the following global gradient estimate, which improves Corollary H above:

Corollary 2.7 ([6]). *Let (X, g) be an n-dimensional complete non-compact Riemannian manifold with $\mathrm{Ric}_g \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $X \times (0,T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{X \times (0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0,T]$,

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leqslant \frac{1}{1+a(\alpha-1)} \cdot \frac{a\alpha^2}{t} + \frac{a\alpha^2 KL}{2(\alpha-1)}, \tag{2.8}$$

where $\alpha > 1$ is any constant.

By integrating the gradient estimate (2.8) along a space-time path, we may obtain the following Harnack inequality:

Corollary 2.9 ([6]). *Let (X, g) be an n-dimensional complete non-compact Riemannian manifold with $\mathrm{Ric}_g \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $X \times (0,T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{X \times (0,T]} v, \quad M := \inf_{X \times (0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, for any $x_1, x_2 \in X$ and $0 < t_1 < t_2 < T$,

$$v(x_1,t_1) \leqslant v(x_2,t_2) \left(\frac{t_2}{t_1}\right)^{\frac{a\alpha}{1+a(\alpha-1)}} \exp\left\{\frac{\alpha \,\mathrm{dist}^2(x_1,x_2)}{4M(t_2-t_1)} + \frac{a\alpha KL}{2(\alpha-1)}(t_2-t_1)\right\},$$

where $\alpha > 1$ is any constant and $\mathrm{dist}(x_1,x_2)$ is the distance between x_1 and x_2.

Theorem 2.10 ([6]). *Let (X, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_g(B_p(2R)) \geqslant -K$ for $K \geqslant 0$. Suppose that $u = u(x,t)$ is a positive solution to (2.1) on $B_p(2R) \times (0,T]$. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{B_p(2R) \times (0,T]} v \quad \text{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $B_p(R) \times (0,T]$,

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leqslant{}& \frac{1}{1+a(\alpha(t)-1)} \cdot \frac{C(n)La\alpha^2(t)}{R^2}\left(\frac{m^2 a\alpha^2(t)}{2(\alpha(t)-1)(m-1)} + 1 + \sqrt{K}R\right) \\ &+ \frac{1}{1+a(\alpha(t)-1)} \cdot \frac{a\alpha^2(t)}{t}, \end{aligned} \tag{2.11}$$

where $\alpha(t) = e^{2KLt}$ and $C(n)$ is a constant depending only on n.

By taking $R \to \infty$ in (2.11), we obtain the following global gradient estimate, which improves Corollary J above:

Corollary 2.12 ([6])**.** *Let* (X, g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g \geqslant -K$ *for* $K \geqslant 0$*. Suppose that* $u = u(x, t)$ *is a positive solution to* (2.1) *on* $X \times (0, T]$*. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{X \times (0,T]} v \quad \textit{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, on $X \times (0, T]$*,*

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leqslant \frac{1}{1 + a(\alpha(t) - 1)} \cdot \frac{a\alpha^2(t)}{t}, \tag{2.13}$$

where $\alpha(t) = e^{2KLt}$*.*

By integrating the gradient estimate (2.13) along a space-time path, we may obtain the following Harnack inequality:

Corollary 2.14 ([6])**.** *Let* (X, g) *be an* n*-dimensional complete non-compact Riemannian manifold with* $\mathrm{Ric}_g \geqslant -K$ *for* $K \geqslant 0$*. Suppose that* $u = u(x, t)$ *is a positive solution to* (2.1) *on* $X \times (0, T]$*. Put*

$$v = \frac{m}{m-1} u^{m-1}, \quad L = (m-1) \sup_{X \times (0,T]} v, \quad M := \inf_{X \times (0,T]} v \quad \textit{and} \quad a = \frac{n(m-1)}{n(m-1)+2}.$$

Then, for any $x_1, x_2 \in X$ *and* $0 < t_1 < t_2 < T$*,*

$$v(x_1, t_1) \leqslant v(x_2, t_2) \exp\left\{ \frac{\alpha(t_2) - \alpha(t_1)}{2KL} \left(\frac{\mathrm{dist}^2(x_1, x_2)}{4M(t_2 - t_1)^2} + \frac{1}{1 + a(\alpha(t_1) - 1)} \cdot \frac{a}{t_1} \right) \right\},$$

where $\alpha(t) = e^{2KLt}$ *and* $\mathrm{dist}(x_1, x_2)$ *is the distance between* x_1 *and* x_2*.*

References

[1] D. G. Aronson, P. Bénilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbb{R}^n$, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), 103-105.

[2] R. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), 113-125.

[3] G. Huang, Z. Huang and H. Li, *Gradient estimates for the porous medium equations on Riemannian manifolds*, J. Geom. Anal. **23** (2013), 1851-1875.

[4] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153-201.

[5] P. Lu, L. Ni, J. Vázquez and C. Villani, *Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds*, J. Math. Pures Appl. **91** (2009), 1-19.

[6] H. Tadano, *Remark on Harnack-type inequalities for the porous medium equation on Riemannian manifolds*, Preprint, 2015.

[7] J. Vázquez, "The Porous Medium Equation", Oxford Mathematical Monographs, 2007.