Conformal length through Laguerre geometry

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1 Extinct points

Definition 1. Let $x\colon I \to \mathbf{R}^n$ be a smooth curve. A point $p \in \mathbf{R}^n \cup \{\infty\}$ is an *extinct point* of $x(t)$ if $(\gamma_p \circ x)^{\cdot\cdot}(t) = 0$, where $\gamma_p \in \mathrm{Conf}(\mathbf{R}^n \cup \{\infty\})$ is a Möbius transformation such that $\gamma_p(p) = \infty$.

If x is a regular curve, then for any t there exists a unique extinct point of $x(t)$ which will be denoted by $\hat{x}(t)$. Using the Clifford multiplication we have $\hat{x}(t) = x(t) - 2\dot{x}(t)\ddot{x}(t)^{-1}\dot{x}(t)$.

Lemma 2. *If $\psi \in \mathrm{Conf}(\mathbf{R}^n \cup \{\infty\})$ and $y = \psi \circ x$, then $\hat{y} = \psi \circ \hat{x}$.*

The *Bernoulli spiral* is given as $x(t) = (e^{\zeta t}, 0) \in \mathbf{C} \times \mathbf{R}^{n-2} = \mathbf{R}^n$, $\zeta \in \mathbf{C} \setminus \{0\}$. A regular curve x satisfies duality condition $x = \hat{\hat{x}}$ if and only if x is Möbius transformation of a Bernoulli spiral.

2 Laguerre sphere space

The *Laguerre sphere space* is $Q_L = \{(C, r) \in \mathbf{R}^n \times \mathbf{R}\} = \mathbf{R}^n \times \mathbf{R}$ with Minkowski metric $|dC|^2 - dr^2$. We think that $S = (C, r) \in Q_L$ represents an oriented $(n-1)$-sphere in $\mathbf{R}^n$ whose center is C, radius is $|r|$ and orientation corresponds to the sign of r. $Q_0 = \{(C, r) \in Q_L |\ r \neq 0\}$ with $g = \frac{1}{r^2}(|dC|^2 - dr^2)$ is isometric to an open dense subset of de Sitter space (Q, g). $Q \setminus Q_0$ is thought of as a set which consists of oriented affine hyperplanes in $\mathbf{R}^n$. Hence $\psi \in \mathrm{Conf}(\mathbf{R}^n \cup \{\infty\})$ naturally induces $\tilde{\psi}\colon Q \to Q$.

Lemma 3. *$\tilde{\psi} \in \mathrm{Isom}(Q)$ for $\psi \in \mathrm{Conf}(\mathbf{R}^n \cup \{\infty\})$.*

3 L-transform

Definition 4. Let $x\colon I \to \mathbf{R}^n$ be a regular curve. Let $S(t) \in Q$ be the sphere which passes through $x(t)$ and $\hat{x}(t)$ perpendicularly to x at $x(t)$ with orientation given by $\dot{x}(t)$. The curve $S\colon I \to Q$ is called the L-transform of $x\colon I \to \mathbf{R}^n$.

Lemma 5. *Let S and T be the L-transforms of x and y respectively. If $\psi \in \mathrm{Conf}(\mathbf{R}^n \cup \{\infty\})$ and $y = \psi \circ x$, then $T = \tilde{\psi} \circ S$.*

Proposition 6. *$S\colon I \to Q$ is L-transform of some regular curve x if and only if there is a past directed lightlike vector field Y along S such that $\nabla_{\dot{S}} Y = 0$ and $g(\dot{S}, Y) = 1$.*

If S is the L-transform of x and $S(t) \in Q_0$, $Y = \dfrac{|\dot{x}|^2}{|\dot{x}|^{\cdot 2}}(\dot{x}, -|\dot{x}|) \in \mathbf{R}^n \times \mathbf{R} \cong T_S Q_0$.

Corollary 7. *Suppose $\dot{S}$ is future directed if $\dot{S}$ is non spacelike. Then by a change of parameter, S has an inverse L-transform time locally.*

4 Schwarzian derivative

The *Schwarizian derivative* of a regular curve $x\colon I \to \mathbf{R}^n$ is defined as $sx = \dddot{x}\dot{x}^{-1} - \frac{3}{2}(\ddot{x}\dot{x}^{-1})^2$, where multiplication is understood to be Clofford multiplication (O. Kobayashi and M. Wada, 2000). $sx(t)$ has a decomposition $sx(t) = sx^{(0)}(t) + sx^{(2)}(t) \in \mathbf{R} \oplus \Lambda^2\mathbf{R}^n$. This quantity sx in substance can be found in a work by K. Yano in 1940. But it seems that Yano was not aware of Clifford algebra nor Schwarzian derivative and he did not reach the scalar part $sx^{(0)}$.

Lemma 8. *$sx^{(0)} = \frac{1}{2}g(\dot{S}, \dot{S})$, where S is the L-transform of x.*

It follows from this that $sx^{(0)}$ is conformally invariant. We also have

Corollary 9. *There is a local parametrization of x such that $\dot{S}$ is lightlike.*

Hence for local theory of curves this parametrization may be useful. Following standard Laguerre differential geometry as in the book by Blaschke in 1929, the Laguerre curvature κ_L and the Laguerre torsion τ_L are computed as follows:

$$\kappa_L = \frac{1}{2}\left(\frac{|\nabla_{\dot{S}}\dot{S}|^{\cdot\cdot}}{|\nabla_{\dot{S}}\dot{S}|^2} - \frac{7}{4}\frac{|\nabla_{\dot{S}}\dot{S}|^{\cdot 2}}{|\nabla_{\dot{S}}\dot{S}|^3}\right), \quad \tau_L = |\nabla_{\dot{S}}\dot{S}|^{1/2}Y - \frac{|\nabla_{\dot{S}}\dot{S}|^{\cdot}}{|\nabla_{\dot{S}}\dot{S}|^{5/2}}\nabla_{\dot{S}}\dot{S} + \frac{1}{|\nabla_{\dot{S}}\dot{S}|^{3/2}}\nabla_{\dot{S}}\nabla_{\dot{S}}\dot{S},$$

where S is the L-transform of x, Y is as in Proposition 6 and parametrization is chosen so that $\dot{S}$ is lightlike.

Proposition 10. *$\kappa_L = \tau_L = 0$ if and only if x is conformal to a Bernoulli spiral with $\Re\zeta^2 = 0$.*

Thus this special Bernoulli spiral is the curve corresponding to what is called *Laguerre cycle.*

5 Closed curves

Parametrization for which the L-transform is lightlike has several nice properties but it is not very much suitable for consideration of closed curves because a lightlike curve cannot be a closed curve in Q. We think of a closed curve as a regular curve $x : [0,1] \to \mathbf{R}^n$ with conditions $x(0) = x(1)$, $\dot{x}(0) = \dot{x}(1)$,...

Definition 11. x is said to be *properly closed* if there exists a parameter $u = u(t) \in [0,1]$ such that for $\tilde{x}$, $\tilde{x}(u) = x(t)$, the L-transform $\tilde{S} : [0,1] \to Q$ is closed and $g(\tilde{S}', \tilde{S}') = \text{const} > 0$, where $' = d/du$. Then we put $l = \sqrt{g(\tilde{S}', \tilde{S}')} > 0$.

I do not yet know any closed curve which is *not* properly closed.

Theorem 12. *For a properly closed curve x, l is uniquely determined.*

Hence l is conformal invariant of x and will be called the *conformal length* of x and denoted by $l(x)$.

Theorem 13. *For any properly closed curve x in $\mathbf{R}^n$, $l(x) \geq 2\pi$. $l(x) = 2\pi$ if and only f x is the circle.*

I hope similar argument will be possible for surfaces or in higher dimensions.

After the talk, Jun O'Hara pointed out that his joint work with R. Langevin and S. Sakara published in Ann. Polonici Math. 108.2 (2013) has something to do with our formulation of Laguerre curvature and torsion.