Indices of isolated umbilics on surfaces

(A joint work with Naoya ANDO and Toshifumi FUJIYAMA).

(GAGT 2015.11.12)

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- Naoya Ando, Toshifumi Fujiyama, Masaaki Umehara, C^1-umbilics with arbitrarily high indices, arXiv:1508.03685 [math.DG].

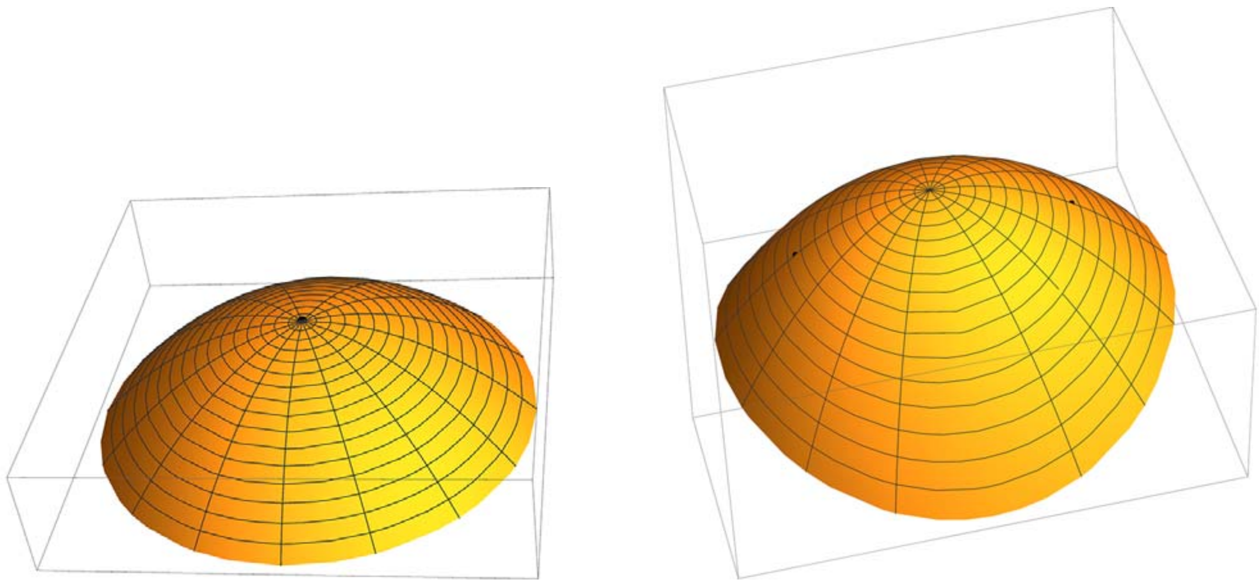


図 1. Elliptic paraboloids $z = x^2 + y^2$ and $z = 2x^2 + y^2$

(1) Indices of vector fields at their isolated zeros
(2) Umbilic points on surfaces in $\mathbf{R}^3$
(3) The history of Caratheodory's conjecture
(4) Examples of C^1-umbilics with arbitrarily high indices

Vector fields on surfaces

Let

$$S \subset \mathbf{R}^3$$

be a surface embedded in $\mathbf{R}^3$. We consider vector fields on S.

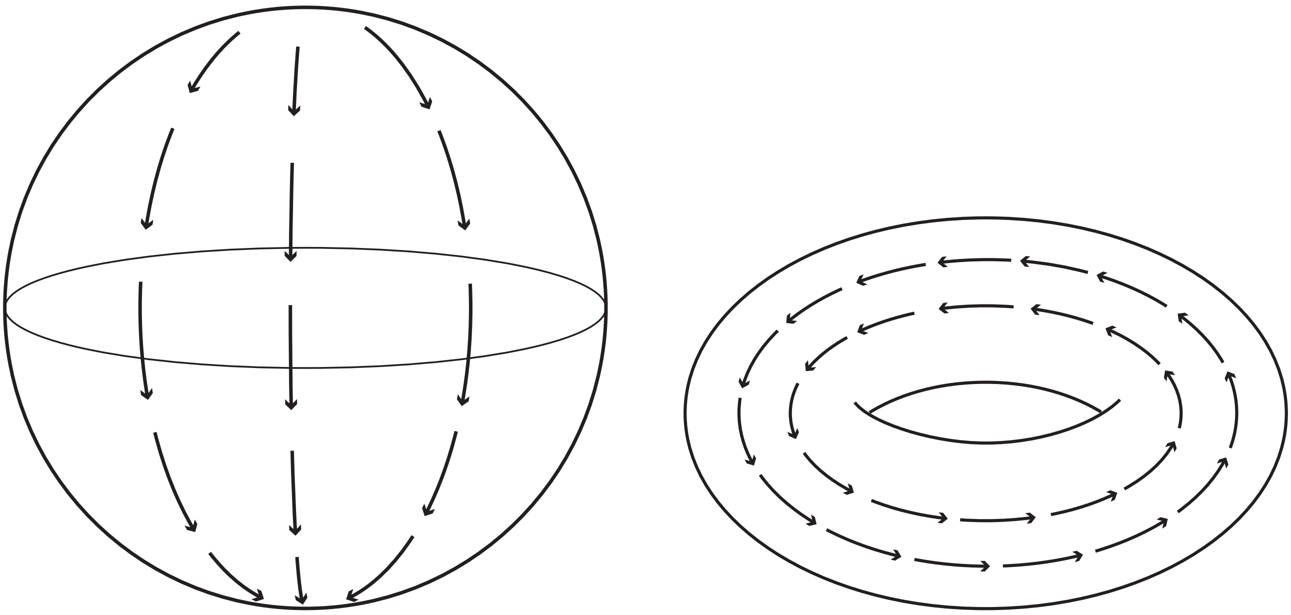


図 2. Vector fields on a sphere and a torus

Taking a local parametrization

$$f : (U^2; u, v) \to S(\subset)\mathbf{R}^3$$

of the surface, we can (locally) indicate the vector fields on the uv-plane.

Indices of vector fields

X : a vector field on $U \subset (\mathbf{R}^2; u, v)$,
p: an isolated zero of X.

$$\gamma(t) := p + \varepsilon(\cos t, \sin t),$$

where ε is a sufficiently small number. Then the winding number of the map

$$S^1 \ni (\cos t, \sin t) \mapsto \frac{X_{\gamma(t)}}{|X_{\gamma(t)}|} \in S^1$$

is called the *index* of the vector field of X at p.

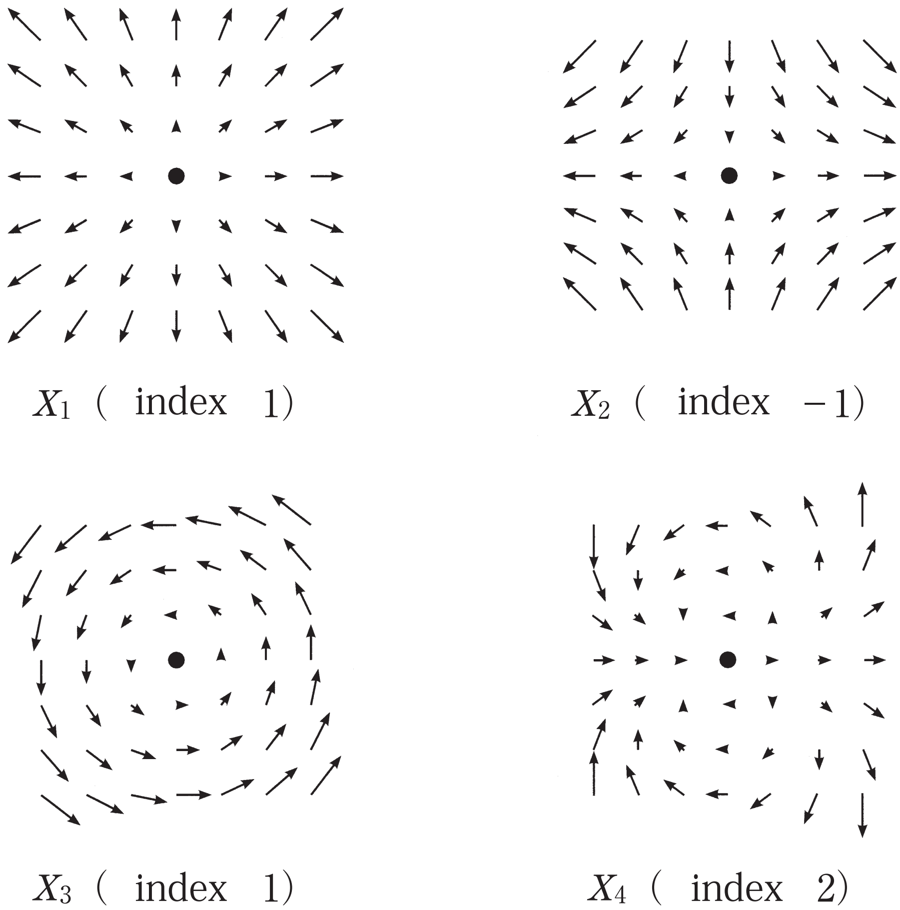


図 3. Vector fields on the uv-plane

The Poincare-Hopf index formula

S: an oriented closed surface

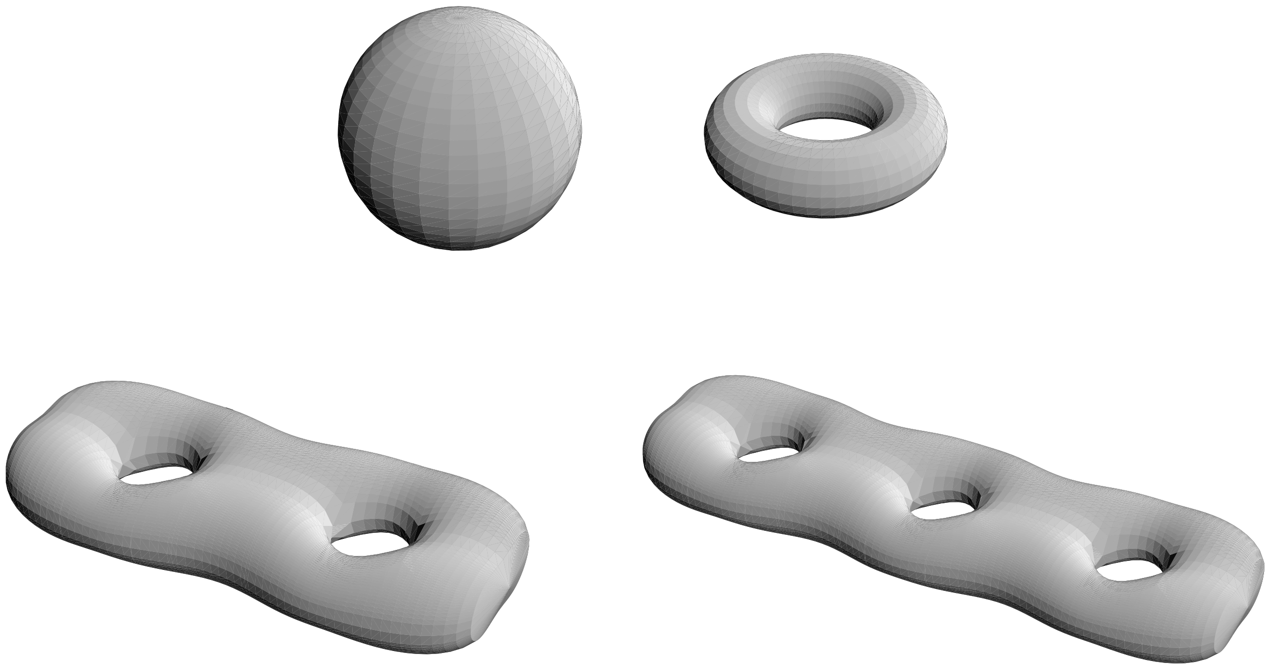


図 4. embedded closed surfaces

X: a vector field on S,
p_1, ..., p_n: the zeros of X on S,
Then the Poincare-Hopf index formula asserts that

$$I(p_1) + \cdots + I(p_n) = \chi(S),$$

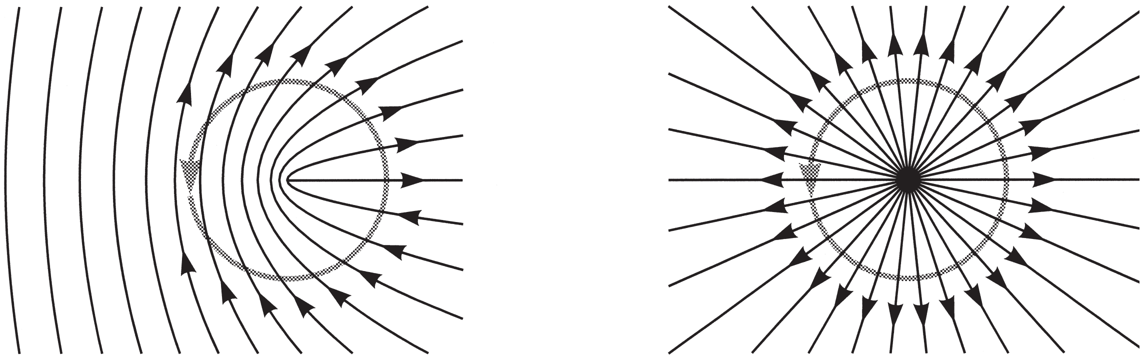
where $I(p_j) = \mathrm{Ind}_{p_j}(X)$ is the index of X at p_j.

Indices of directional vector fields

Instead of vector field, we consider an assignment

$$S \ni p \mapsto L_p \ : \text{ a line passing through } p$$

called a *direction field*. It induces a flow without orientation on the surface.



$a<b<c$ （index 1/2）　　　$a=b<c$ （index 1）

図 5. The curvature line flows of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$

On can compute the index at an isolated singular point of a given direction field as same as the case of vector fields.

It takes values in *half-integers*, in general.

Examples of direction fields

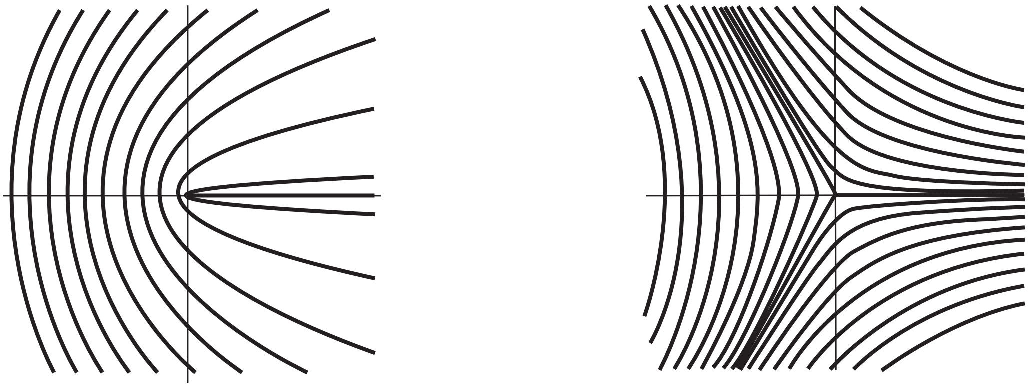


図 6. Examples of direction fields

The above figure indicates the eigen flows of symmetric matrices

$$A = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}, \qquad B = \begin{pmatrix} u & -v \\ -v & -u \end{pmatrix}.$$

Let

$$p_1, \ \ldots, \ p_n$$

be the set of isolated singular points of the direction field of the closed surface S. We denote by $I(p_j)$ the index at p_j. Then, it holds that

$$I(p_1) + \cdots + I(p_n) = \chi(S),$$

which is the Poincare-Hopf index formula for direction fields.

Umbilics on surfaces

$U \subset (\mathbf{R}^2; u, v)$; a domain,
$f : U \to \mathbf{R}^3$; an immersion,

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|}$$

is a unit normal vector field on U. The three functions

$$E := f_u \cdot f_u, \quad F := f_u \cdot f_v, \quad G := f_v \cdot f_v.$$

are called the coefficients of the first fundamental form. Also, the coefficients of the second fundamental form are given by

$$L := f_{uu} \cdot \nu, \quad M := f_{uv} \cdot \nu, \quad N := f_{vv} \cdot \nu.$$

Then the eigenvalues of the matrix

$$A_f := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

are principal curvatures, and their eigenvectors correspond to the principal directions.

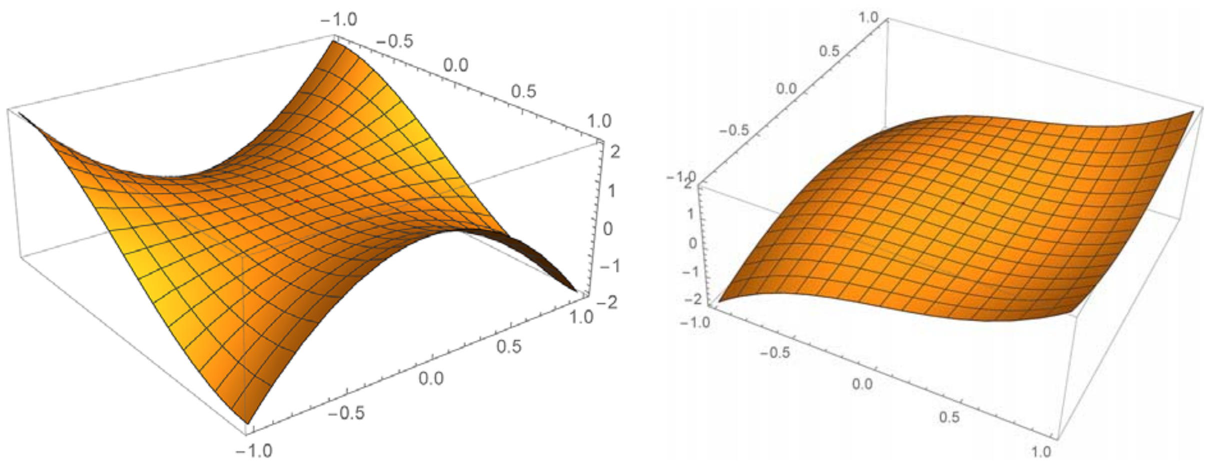


図 7. The graphs of $z = u^3 - 3uv^2 (= \mathrm{Re}(z^3))$ and $z = u^3 + uv^2 (= \mathrm{Re}(z^2\bar{z}))$

The eigenflow of the matrix A_f is called the *curvature line flow*. The umbilics correspond to the diagonal points of A_f.

Loewner's conjecture

By the Poincare-Hopf index formula,

- the total sum of indices of all umbilics on a convex surface is equal to 2.

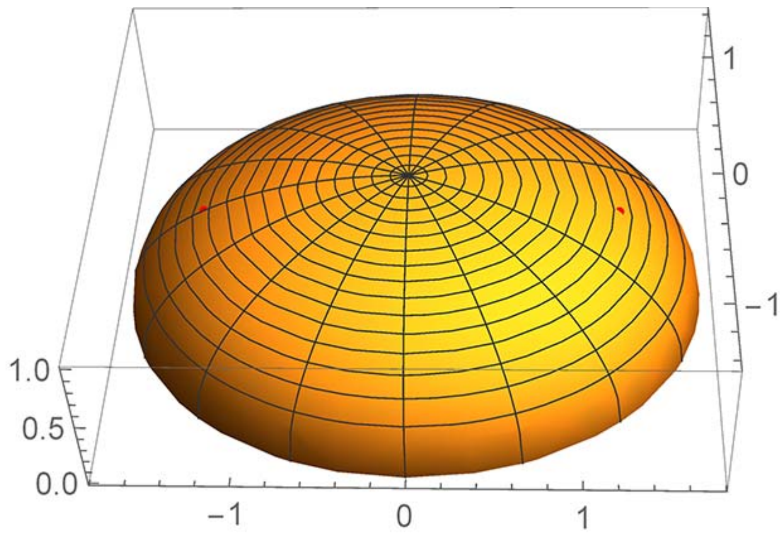


図 8. The upper half of $3x^2 + 2y^2 + z^2 = 1$

- Is the number of umbilics on a convex surface greater than or equal to 2? (Caratheodory's conjecture).
- Is the indices of umbilics on a regular surface less than or equal to one? （Loewner's conjecture）.

History

- H. Hamburger, Beweis einer Caratheodoryschen Vermutung I, Ann. Math. 41, (1940)63–86.
- H. Hamburger, *Beweis einer Caratheodoryschen Vermutung II*, Acta Math. 73, (1941) 175–228.
- H. Hamburger, *Beweis einer Caratheodoryschen Vermutung III*, Acta Math. 73, (1941) 229–332.
- G. Bol, *Über Nabelpunkte auf einer Eifläche*, Math. Z. 49, (1944) 389–410.
- T.-K. Milnor, *On G. Bol's proof of Carathodory's conjecture*, Commun. Pure Appl. Math. 12, (1959) 277–311.
- C. J. Titus, *A proof of a conjecture of Loewner and of the conjecture of Carathodory on umbilic points*, Acta Math. 131, (1973), 43–77.
- H. Scherbel, *A new proof of Hamburger's index theorem on umbilical points*, Thesis 10281 (1993), ETH Zürich.
- V. Ivanov, *The analytic Caratheodory conjecture*, Sib. Math. J. 43, (2002) 251–322.
- B. Guilfoyle and W. Klingenberg, *Proof of the Carathodory Conjecture*, (2013), preprint.

Bates' example

- Larry Bates, *A weak counterexample to the Carathéodory conjecture*, Differential Geometry and its Applications, **15** (2001) 79–80.

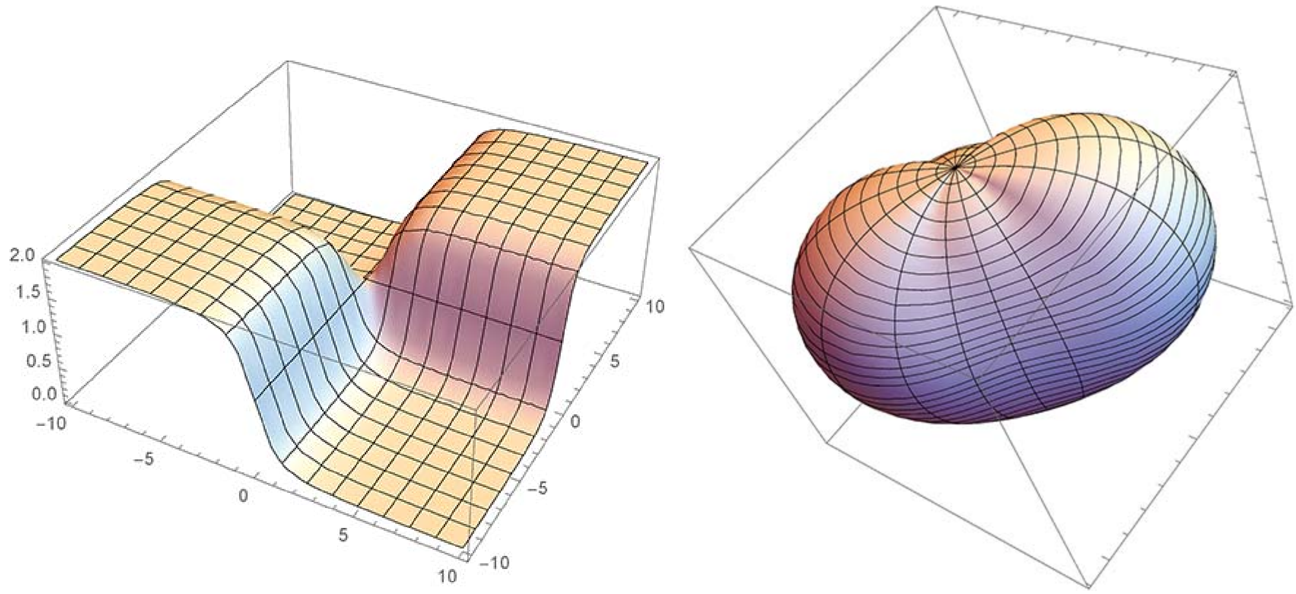


図 9. Bates' graph and its inversion

The image of the graph

$$B(x,y) := 2 + \frac{xy}{\sqrt{1+x^2}\sqrt{1+y^2}}$$

has no umbilics, so its inversion

$$f := F/|F|^2, \qquad F := (x, y, B(x,y))$$

is a closed regular surface, which is differentiable but not C^1-regular at $(0,0,0)$. This implies that $(0,0,0)$ is a differentiable umbilic of index 2. (cf. Bachelor's thesis of Fujiyama).

(**Advantages of the inversion**):

(1) A conformal transformation preserves the umbilic flow.
(2) It maps to spheres to spheres.
(3) It is useful to construct non-analytic smooth surfaces.

The regularity of surfaces after inversion

Let U be a domain containing $(0,0)$, and

$$f : \mathbf{R}^2 \setminus U \to \mathbf{R} \setminus \{0\}$$

a smooth function.

Theorem 1 (Fujiyama-Ando-U)**.** *If f/r is bounded, and*

$$\left| \frac{f^2 - 2rff_r}{r^2} \right| < 1, \tag{1}$$

then the inversion of the image of f can be expressed as a continuous graph $Z = Z_f(X,Y)$ near the origin $(0,0,0)$, where

$$r := \sqrt{x^2 + y^2}.$$

Moreover, under the assumtion (1)*, $Z = Z_f(X,Y)$ is differentiable at the origin if and only if*

$$\lim_{r\to\infty} \frac{f}{r} = 0. \tag{2}$$

The Bates' function is bounded and satisfies (1) and (2). So the inversion of the Bates' function induces a differentiable umbilic.

The regularity of surfaces after inversion II

About C^1-regularity, we have the following:

Proposition 2. *If the function f is bounded, and*

$$\text{(a)} \quad \lim_{r\to\infty} f_r = 0, \qquad \text{(b)} \quad \lim_{r\to\infty} \frac{f_\theta}{r} = 0,$$

then the inversion of f can be expressed as a graph $z = Z_f(X, Y)$ which has C^1-regularity at the origin $(0, 0, 0) = (0, 0, Z_f(0, 0))$.

The Bates' function does not satisfy (b). In fact, the unit normal vector field

$$\nu_B := \frac{(-B_x, -B_y, 1)}{\sqrt{1 + B_x^2 + B_y^2}}$$

satisfies

$$\lim_{x\to\infty} \nu_B(x, 0) = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),$$
$$\lim_{y\to\infty} \nu_B(0, y) = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}).$$

Identifiers of umbilics

For a given function

$$f : (U, x, y) \to \mathbf{R},$$

we define the following vector field on U by

$$D_f := d_1 \frac{\partial}{\partial x} + d_2 \frac{\partial}{\partial y},$$
$$d_1 := (1 + f_x^2) f_{xy} - f_x f_y f_{xx},$$
$$d_2 := (1 + f_x^2) f_{yy} - f_{xx}(1 + f_y^2).$$

Theorem 3. *The vector field D_f on U has the following properties:*

(1) $D_f(x_0, y_0) = \mathbf{0}$ *if and only if $P := (x_0, y_0) \in U$ is an umbilic. (cf. Ghomi-Howard 2012)*.

(2) *Moreover, the number*

$$\mathrm{Ind}_P(D_f)/2$$

is equal to the index of curvature line flow at P.

Using D_f, we can easily find four umbilics on an ellipsoid. The function d_1 for Bates' function is computed by

$$d_1 = \frac{x^6\left(y^2+1\right) + 3x^4\left(y^2+1\right) + x^2\left(6y^2+3\right) + 2y^2 + 1}{\left(x^2+1\right)^{9/2}\left(y^2+1\right)^{5/2}}.$$

This implies that B has no umbilics.

Polar identifier for umbilics

Let $f : (U, 0) \to \mathbf{R}$ be a smooth function.
We set

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and define a new vector field by

$$\Delta_f := \delta_1 \frac{\partial}{\partial x} + \delta_2 \frac{\partial}{\partial y},$$
$$\delta_1 := -f_\theta \left(1 + f_r^2 + r f_r f_{rr}\right) + r\left(1 + f_r^2\right) f_{r\theta},$$
$$\delta_2 := \left(1 + f_r^2\right)\left(r f_r + f_{\theta\theta}\right) - f_{rr}\left(r^2 + f_\theta^2\right).$$

Theorem 4 (Ando-Fujiyama-U)**.**
The vector field Δ_f satisfies the following properties:
(1) $\Delta_f(P) = \mathbf{0}$ *if and only if $P \in U$ is an umbilic.*
(2) *If $P = (0, 0)$, then the number*

$$1 + \frac{\mathrm{Ind}_P(\Delta_f)}{2}$$

gives the index of curvature line flow of the graph of f.

Example 1. The function

$$f = x^3 - 3xy^2 = r^3\cos\theta(-1 + 2\cos 2\theta)$$

has an index $-1/2$ at $(0, 0)$, which follows from

$$\Delta_f = \left(-6r^3 \sin 3\theta, -6r^3\left(9r^4 + 2\right)\cos 3\theta\right).$$

Example 2. The function

$$f = x^3 + xy^2 = r^3\cos\theta$$

has an index $1/2$ at $(0, 0)$, which follows from

$$\Delta_f = \left(-2r^3 \sin\theta, -2r^3 \cos\theta\left(2 - 3r^4 - 6r^4\cos 2\theta\right)\right).$$

The main theorem

We set

$$U_1 := \{(x, y) \in \mathbf{R}^2 \,;\, x^2 + y^2 < 1\}.$$

Theorem (Ando-Fujiyama-U.) For each positive integer, there exists a C^1-differentiable immersion

$$\varphi : U_1 \to \mathbf{R}^3$$

satisfying the following properties:

(1) φ is real analytic on $U_1^* := U_1 \setminus \{(0, 0)\}$.

(2) The index of curvature line flow of $\varphi(x, y)$ at $(0, 0)$ is equal to $1 + \dfrac{m}{2}$.

In fact, we consider the function

$$(f =) f_m(r, \theta) := 1 + \tanh(r^a \cos m\theta),$$
$$(0 < a < 1,\ m = 1, 2, 3, \cdots).$$

Then the inversion of it satisfies the above conditions (1) and (2).

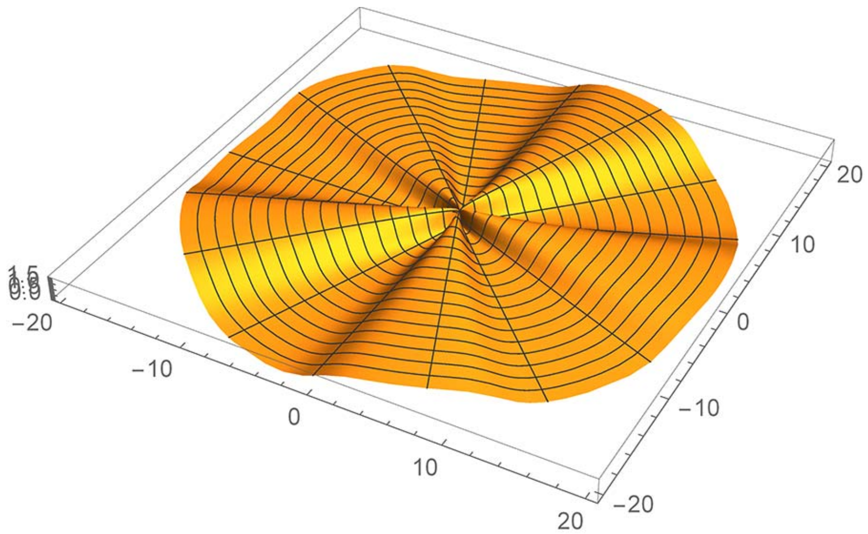


図 10. The image of f for $m = 5$ and $a = 1/5$.

The proof of the theorem

(The surface obtained by the inversion of f_5)

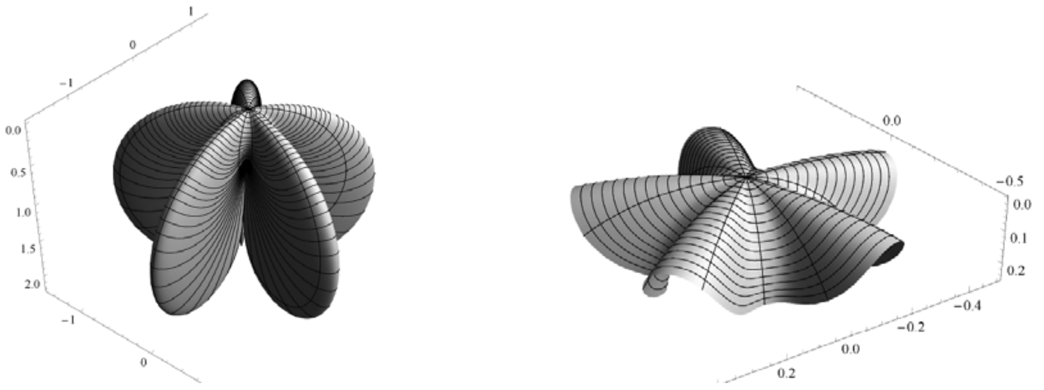


図 11. The inversion of f for $m = 5$ and $a = 1/5$.

We set

$$f = 1 + F(r^a \cos m\theta), \qquad F(x) := \tanh(x).$$

Then

(1) $F(-x) = -F(x)$,
(2) $F'(x) > 0$ for $x \in \mathbf{R}$,
(3) $F''(x) < 0$ for $x > 0$,
(4) $F(x) \sim 1 - e^{-2x}$ for $x >> 0$.

$$\delta_1 = -mr^a s_m \Big(a r^a c_m F''(r^a c_m) + (a-1) F'(r^a c_m) \Big)$$

and

$$\begin{aligned} r^{2-3a}\delta_2 = & -r^{2-a}\left(a^2 c_m^2 - m^2 s_m^2\right) F''(c_m r^a) \\ & + a c_m \left(a^2 c_m^2 - a m^2 + m^2 s_m^2\right) F'(c_m r^a)^3 \\ & - c_m r^{2-2a}\left(a^2 - 2a + m^2\right) F'(c_m r^a), \end{aligned}$$

where $c_m = \cos m\theta$, $s_m = \sin m\theta$.

$$\mathrm{Ind}_\infty(\Delta_f) = -m,$$

$$I_f(\infty) = 1 - \frac{m}{2},$$

The index after inversion $= 2 - I_f(\infty) = 1 + \dfrac{m}{2}$.

An alternative proof without inversion

We set

$$\lambda := r^2 \tanh(r^{-a} \cos\theta) \qquad (0 < a < 1),$$

where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The the index of H_λ is equal to

$$1 + \frac{m}{2}.$$

We set

$$\nu := \frac{1}{1+\lambda_x^2+\lambda_y^2}\Big(2\lambda_x, 2\lambda_y, \lambda_x^2+\lambda_y^2-1\Big).$$

Then it is a unit normal vector of the surface

$$P = (x, y, \lambda) - \lambda\nu,$$

which is C^1-differentiable at $(0,0)$ and its curvature line flow has the index

$$1 + \frac{m}{2}.$$

In fact, λ is related to f_m by

$$\hat{\lambda} := (x^2+y^2)^{-1}\lambda\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$
$$(= \tanh(r^a \cos(m\theta))) = f_m,$$

and the formula

$$\mathrm{ind}_0(H(\mu)) + \mathrm{ind}_\infty(H(\hat{\mu})) = 2$$

holds for an arbitrary given C^∞-function $\mu : \mathbf{R}^2 \setminus \{(0,0)\} \to \mathbf{R}$.