Webs, Foams and Instantons.
Tokyo

Tom Mrowka

MIT

Oct 28-30,2014

The goal of this talk explain how two threads in knot theory tie together.

Instanton Floer homology (originally constructed by Floer in the late 1980's building on work of Uhlenbeck, Taubes and Donaldson). A generalization to knots was implicit in work Kronheimer and I did in in 1992-3. This has become more explicit in the past 5 years and has some interesting applications to knot theory. Related or inspiring symplectic constructions where carried out by Seidel-Smith and Wehrheim-Woodward.

Khovanov Homology and Khovanov-Rozansky homology, and theory of Webs and Foams. Work of Stoisic, Vaz and Kapustin-Li etc... Combinatorially defined invariants of knots coming from representation theory.

Given some extra data like a compact Lie group $G = SU_N$ we assign to a three manifold Y a vector space over $\mathbb{C}$, or $\mathbb{Z}$-module etc:

$$\mathbb{I}(Y)_N$$

Often we'll drop N from the notation!
To four-dimensional cobordisms X with $\partial X = -Y_0 \coprod Y_1$ we assign maps:

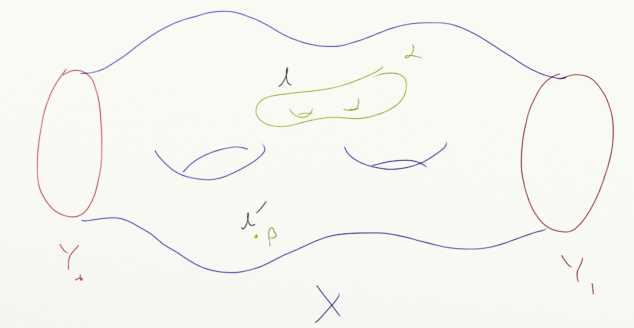
$$\Phi_X : \mathbb{I}(Y_0) \to \mathbb{I}(Y_1)$$

More generally given a homology class $\alpha \in H_*(X)$ and an integer $0 \leq \ell \leq N$ we get

$$\Phi_X(\alpha)_\ell : \mathbb{I}(Y_0) \to \mathbb{I}(Y_1)$$

so that

$$\Phi_X(\alpha)_\ell \Phi_X(\beta)_{\ell'} = (-1)^{\|\alpha\|\|\beta\|} \Phi_X(\beta)_{\ell'} \Phi_X(\alpha)_\ell$$



Some Basic Feature of 3-manifold Topology

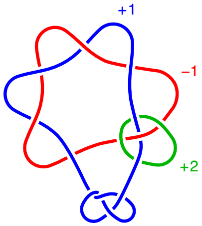
Examples of three manifolds.

$$S^3, S^1 \times S^2, T^3, S^1 \times \Sigma_g$$

A fibered 3-manifold

$$S^1 \times_\tau \Sigma_g$$

where $\tau : \Sigma_g \to \Sigma_g$ is a diffeomorphism. Surgery on knot or link (this is now a general example.)



The Thurston norm.

$$\| \cdot \|_T : H_2(Y;\mathbb{Z}) \to \mathbb{R}$$

$\| \cdot \|_T$ measure the complexity of surfaces representing a homology class.

- If Σ is connected oriented surface define

$$x(\Sigma) = \min\{0, -\chi(\Sigma)\}$$

- If Σ is disconnected (oriented) define $x(\Sigma)$ to be the sum of x of its components.
- $\|\alpha\|_T$ is the minimum of $x(\Sigma)$ where $\Sigma \hookrightarrow Y$ *represents* α.

Instanton Floer Homology

Chern-Weil theory in dimension 4 and Anti-Self-Duality. Let $P \to X$ be a principal SU_2-bundle and let E denote the associated vector bundle.

$$-\frac{1}{8\pi^2}\int_X tr(F_A \wedge F_A) = \langle c_2(E), [X] \rangle.$$

Thus 2-forms have a conformally invariant decomposition.

$$\begin{array}{ccccc} \Lambda^2 & = & \Lambda^+ & \oplus & \Lambda^- \\ & & *=1 & & *=-1 \end{array}$$

Explicitly: e^1, e^2, e^3, e^4 oriented orthonormal frame.

$$\omega_I = e^1\wedge e^2 + e^3\wedge e^4, \omega_J = e^1\wedge e^3 - e^2\wedge e^4, \omega_K = e^1\wedge e^4 + e^2\wedge e^3$$

span Λ^+, the Self-Dual two forms and

$$e^1 \wedge e^2 - e^3 \wedge e^4,\; e^1 \wedge e^3 + e^2 \wedge e^4,\; e^1 \wedge e^4 - e^2 \wedge e^3$$

span Λ^-, the Anti-Self-Dual two forms. Note that Λ^+ and Λ^- are pointwise orthogonal under both the riemmanian inner product and the wedge product.

Thus we can decompose the curvature of a connection:

$$F_A = F_A^+ + F_A^-.$$

$F_A^+ = 0 \quad (F_A^- = 0)$ are the Anti-Self-Dual (Self-Dual) Yang-Mills equation.
Note that the Bianchi identity implies

$$d_A F_A = 0$$

and hence if A is SD or ASD we have

$$d_A^* F_A = - * d_A * F_A = \pm * d_A F_A = 0.$$

Thus SD and ASD are critical points for E but more is true!

If X is a closed four-manifold recall:

$$8\pi^2 k = \int_X tr(F_A \wedge F_A), \quad E(A) = -\int_X tr(F_A \wedge *F_A) \geq 0.$$

Thus adding these formulae gives:

$$E(A)+8\pi^2 k = \int_X tr(F_A \wedge (-*F_A + F_A)) = -2\int_X tr(F_A^- \wedge * F_A^-) \geq 0$$

while subtracting gives

$$E(A) - 8\pi^2 k = -2\int_X tr(F_A^+ \wedge *F_A^+) \geq 0$$

Thus

$$E(A) \geq 8\pi^2 |k|$$

with equality if and only if $F_A^+ = 0$ when $k \geq 0$ and $F_A^- = 0$ when $k \leq 0$.

The basic instanton. We'll use quaternionic notation.
$\mathbb{H} = \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ where $IJ = K$ + cyclic and $IJ = -JI$ + cyclic.

$$x = x_0 + x_1 I + x_2 J + x_3 K$$

Conjugation

$$x = x_0 - x_1 I - x_2 J - x_3 K.$$

$$S^7 \subset \mathbb{H}^2 \text{ the unit sphere}$$

Two different $SU_2 = Sp(1) = \{\mathbf{x} \in \mathbb{H} | \bar{\mathbf{x}}\mathbf{x} = \mathbf{1}\}$ actions.

$$(x, y)q = (xq, yq) \text{ or}$$
$$(x, y)q = (\bar{q}x, \bar{q}y).$$

In either case the quotient is $S^4 = \mathbb{HP}^1$.

Thus we have two principal bundles with total space S^7.

$$P_\pm \to S^4$$

where P_+ has the right action and P_- has the left action. These are the unit sphere bundles of $\mathbb{H}$-bundles $S_\pm \to S^4$. For example

$$S_+ = \{([x, y], (v_0, v_1)) | (v_0, v_1) = \bar{h}(x_0, x_1), \quad h \in \mathbb{H}\}.$$

Give $P_\pm$ the connection $A_\pm$ which declares that the horizontal space at $p \in P_\pm$ is the orthogonal complement of the fiber $P_\pm \to S^4$. $Sp_2 (= Spin_5)$ acts on S^7 isometrically preserving the connections $A_\pm$. The stabilizer $Stab_p$ of the point $p = (0, 1) \in S^7 \subset \mathbb{H}^2$ is a copy of $Sp_1 (= SU_2 = Spin_3)$ of matrices of the form

$$\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$$

The curvature of the connection is an SU_2 equivariant map

$$\Lambda^2(T_p^* S^4) = \Lambda^+ \oplus \Lambda^-(T_p^* S^4) \to \mathrm{ad} P_\pm|_p.$$

At $(0, 1) \in S^7$ we can write

$$T_{(1,0)} S^7 = \mathbb{H} \oplus \mathrm{im}\mathbb{H}.$$

The horizontal space is $\mathbb{H} \oplus 0$ and the vertical tangent space at p is identified with $0 \oplus \mathrm{im}\mathbb{H}$. $Stab_p$ acts trivially on the vertical tangent space and by left multiplication on P_+ and right multiplication on P_-.

We need to understand the SU_2 action. Write $dx = dx_0 + dx_1 I + dx_2 J + dx_3 K$. Note that $dx \wedge d\bar{x}$ is a purely imaginary 2-form ($\mathrm{Im}\mathbb{H}$-valued). Indeed

$$dx \wedge d\bar{x} = -2(\omega_I I + \omega_J J + \omega_K K).$$

In particular $dx \wedge d\bar{x}$ is self-dual.

$$d\bar{g}xh \wedge d\overline{\bar{g}xh} = \bar{g}dx \wedge d\bar{x}g.$$

Thus $dx \wedge d\bar{x}$ is invariant under the left action (and equivariant under the right action) so the left action is trivial on self-dual forms. Since Thus A_+ is a self-dual connection.
Similarly $d\bar{x} \wedge dx$ is anti-self-dual and invariant under the left action and equivariant under the right action. Thus A_- is an anti-self-dual connection.

The connection A_+ at the point $(x, y) \in S^7 \subset \mathbb{H}^2$ is

$$\mathrm{Im}(\bar{x} \wedge dx + \bar{y} \wedge dy).$$

We can trivialize P_+ (or P_-) by the section

$$x \mapsto \frac{(x, 1)}{(1 + |x|^2)^{\frac{1}{2}}}$$

Show that the connection one-form

$$\begin{aligned} a &= \frac{\mathrm{Im}\bar{x} \wedge dx}{(1 + |x|^2)} \\ &= \frac{\bar{x} \wedge dx - x \wedge d\bar{x}}{2(1 + |x|^2)} \end{aligned}$$

represents A_+ in a suitable trivialization of P_+.

We can construct from A_+ other ASD-connections using conformal invariance. The dilatation $\tau_\lambda(x) = \lambda x$ induces a conformal diffeomorphism. Indeed the basic instanton is invariant under SO_5 acting on S^4 but the conformal group $SO_{5,1}$ acts so effectively there is a

$$SO_{5,1}/SO_5 = \mathbb{H}^5$$

worth of ASD-connections in P_+. Atiyah-Hitchin-Singer prove any ASD connections in P_+ is gauge equivalent to one of these.

This example exhibits the phenomenon of bubbling. As the the parameter $\lambda \mapsto \infty$ the connections

$$\tau_\lambda^*(a) = \frac{\mathrm{Im}\lambda\bar{x} \wedge d\lambda x}{(1 + |\lambda x|^2)}$$
$$= \frac{\mathrm{Im}\bar{x} \wedge dx}{(1/\lambda^2 + |x|^2)}$$

converge away from the origin to

$$\frac{\mathrm{Im}\bar{x} \wedge dx}{(|x|^2)}$$

which is gauge equivalent to zero! by the gauge transformation

$$g(x) = x/|x|.$$

The Chern-Simons Functional. $Q = Y \times SU_2 \to Y$ a principal SU_2 bundle. Suppose that $Y = \partial X$ and that there is an SU_2-bundle $P \to X$. B is a connection in Q extending to a connection A in P. The Chern-Weil integral

$$\int_X tr(F_A \wedge F_A) \pmod{8\pi^2\mathbb{Z}}$$

does not depend on A!. This the Chern-Simons invariant of B. Let Γ be the connection coming for the trivialization and write $B = \Gamma + b$. Consider the connection $A = \Gamma + tb$ on $[0,1] \times Q \to [0,1] \times Y$. Then

$$F_A = dt \wedge b + tdb + t^2 b \wedge b$$

then

$$CS(B) = 2\int_{[0,1]\times Y} dt \wedge tr(tb \wedge db + t^2 b \wedge b \wedge b)$$

or

$$CS(B) = \int_Y tr(b \wedge db + \frac{2}{3} b \wedge b \wedge b).$$

$$CS(B) = \int_Y tr(b \wedge db + \frac{2}{3} b \wedge b \wedge b).$$

The first variation:

$$\frac{d}{dt} CS(B + tc) = 2 \int_Y tr(c \wedge db + c \wedge b \wedge b) = 2 \int_Y (c \wedge F_B).$$

So critical points are flat connections $F_B = 0$ and critical points. The automorphism group upto gauge transformation equivalent to representation $\rho : \pi_1(Y) \to SU_2$ **upto** conjugacy.

The downward gradient flow equation

$$\frac{dB}{dt} = - * F_B$$

is equivalent to the Anti-Self-Dual Yang Mills Equations for A the connection on $\mathbb{R} \times Y$ induced from $B(t)$

$$F_A = - *_4 F_A$$

$$F_A = dt \wedge \frac{dB}{dt} + F_B = - *_4 (F_B + dt \wedge \frac{dB}{dt}) = -dt \wedge *F_B - *\frac{dB}{dt}.$$

The Hessian of CS. B is flat the linearization of the equation $*F_B = 0$ is the operator $*d_a : \Omega^1(Y, \mathrm{ad}P) \to \Omega^1(Y, \mathrm{ad}P)$, the Hessian of CS.

Floer used rep mods conjugacy as generators of a complex and moduli spaces of ASD connections on $\mathbb{R} \times Y$ to construct the differential.

A major miracle occurs here. If we view a solution $B(t)$ as a connection A on $\mathbb{R} \times Y$ the connection satisfies the Anti-Self-Dual Yang-Mills equation

$$F_A = - *_4 F_A.$$

As a $4d$-connection

$$F_A = dt \wedge \frac{dB}{dt} + F_B$$

and

$$*F_A = dt \wedge *_3 F_B + *_3 \frac{dB}{dt}$$

This later equations is invariant not just under $\mathcal{G}_3 = \{g : Y \to SU_2\}$ but under $\mathcal{G}_4 = \{h : \mathbb{R} \times Y \to SU_2\}$.

Problems.

- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.
- $\mathcal{B}$ is singular.
- CS is not single valued (cf Novikov homology).
- Compactification of moduli spaces (including gluing).
- Construct pertubations to achieve Morse-Smale Condition that do no destroy compactness properties.

The gauge group $\mathcal{G} = \{g : Y \to SU_2\}$ acts on $\Omega^1(Y) \otimes su_2$ by

$$g \cdot b = gbg^{-1} + gdg^{-1}.$$

Under this action we have

$$CS(g \cdot B) = CS(B) + \deg(g) \quad \text{and} \quad F_{g \cdot B} = gF_B g^{-1}$$

Thus the set of critical points of CS is preserved by the $\mathcal{G}$ action and

$$\{a | F_B = 0\} = Hom(\pi_1(Y), SU_2)/\mathrm{conj} = R(Y)$$

where the identification is by the holonomy representation.

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$
- $S^1 \times S^2$. $R(Y) = SU_2/\mathrm{conj}$ an interval.
- T^3. $R(Y) = S^1 \times S^1 \times S^1/\{\mathbb{Z}_2\}$.
- (Fintushuel-Stern) $S(p, q, r)$ a Seifert fibered homology sphere with three exceptional fibers. The representation space is a finite set of isolated points.
- (Fintushuel-Stern) S a Seifert fibered homology sphere with four exceptional fibers. The representation space is a union of S^2's

Notice that a reducible representation into SU_2 lives in a U_1-subgroup. We can understand $Hom(\pi_1(Y), U_1) = H^1(Y; U_1)$ via the long exact sequence

$$H^1(Y, \mathbb{Z}) \to H^1(Y; \mathbb{R}) \to H^1(Y; U_1) \to H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{R})$$

Thus

$$H^1(Y; U_1) \equiv U_1^{b_1} \times \mathrm{Tor}(H^2(Y, \mathbb{Z})).$$

Thus for a three manifold $H^1(Y; U_1) = \{0\}$ if and only if $H_1(Y; \mathbb{Z}) = 0$, i.e. Y is a homology sphere.

Here is Fintushel and Stern's example. Recall that $S(p, q, r)$ is the intersection of

$$V(p, q, r) = x^p + y^q + z^r = 0 \cap S^5 \subset \mathbb{C}^3.$$

This a Seifert fibered space with the S^1-action

$$u \cdot (x, y, z) = (u^{qr} x, u^{pr} y, u^{qp} z).$$

The quotient is an orbifold S^2 with three orbifold points with orders p, q and r.

For a global quotient like this we can form the homotopy quotient

$$(S(p,q,r) \times ES^1)/S^1 = S(p,q,r)//S^1$$

The fundamental group of $S(p,q,r)//S^1$ is called the orbifold fundamental group and has a presentation in this case as follows.

$$\pi_1(S(p,q,r)//S^1) \equiv \{t,u,v | t^p = 1, u^q = 1, v^r = 1, tuv = 1\} = T_{p,q,r}$$

Furthermore the long exact homotopy sequence looks like

$$\to \pi_1(S^1) \to \pi_1(S(p,q,r)) \to \pi_1(S(p,q,r)//S^1) \to 1.$$

or

$$\to \mathbb{Z} \to \pi_1(S(p,q,r)) \to T_{p,q,r} \to 1.$$

Thus $\pi_1(S(p,q,r))$ has a presentation

$$\{t,u,v,h | t^p = h^{-b_1}, u^q = h^{-b_2}, v^r = h^{-b_3}, tuv = h^b\}$$

for some integers b, b_1, b_2, b_3. The data $(b, p/b_1, q/b_2, r/b_3)$ are called the Seifert invariants.

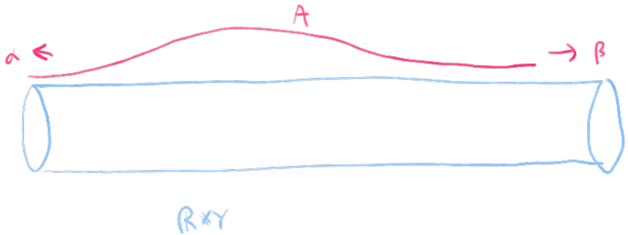
For $\rho : \pi_1(S(p,q,r)) \to SU_2$ is a representation write $\tilde{T}, \tilde{U}, \tilde{V}$ and $\tilde{H}$ for the images of the generators. Then H commutes with the others so if ρ is not trivial $\tilde{H} = \pm 1$. Then the other matrices satisfy the equations

$$\tilde{T}^p = \pm 1, \tilde{U}^q = \pm 1, \tilde{V}^r = \pm 1, \tilde{T}\tilde{U}\tilde{V} = \pm 1.$$

In particular pushing the representation into SO_3 they give rise to a representation of the triangle group $T_{p,q,r}$. These representation we can understand rather easily. Let T, U and V denote the images in SO_3 of $\tilde{T}$ etc. Then T, U, V are rotations about axes ℓ_T, ℓ_U and ℓ_V by angles $2\pi k/p, 2\pi l/q$ and $2\pi m/r$. The fact that these three element satisfy $TUV = 1$ implies that the the products of these rotations is the identity.

Given a pair of critical points ρ, σ set

$$M(\rho,\sigma) = \{A \mid -\int_Z tr(F_A \wedge *F_A) < \infty,\ F_A = -*F_A,$$
$$\lim_{t\mapsto-\infty}[A|_{t\times Y}] = \rho,\ \lim_{t\mapsto\infty}[A|_{t\times Y}] = \sigma\}/\mathcal{G}_4.$$



Note that A is ASD if and only if

$$\|F_A\|_L^2(\mathbb{R}\times Y) = -\int_Z tr(F_A \wedge *F_A) = " - CS(\rho) + CS(\sigma)"$$

where the difference is computed with respect the path $B(t) = A|_{t\times Y}$.
Compare to finite dimensional Morse theory. We have the identity

$$E_f(x) = \frac{1}{2}\int_a^b (|\frac{dx}{dt}|^2 + |\nabla_x f|^2)dt = -f(a)+f(b)+\frac{1}{2}\int_a^b (|\frac{dx}{dt} + \nabla_x f|^2)dt$$

whence $E_f(x) = -f(a) + f(b)$ if and only if x is a downward gradient flow line

$$\frac{dx}{dt} = -\nabla_x f.$$

Given $B_{\pm}$ flat connections with central stabilizer choose a path $B(t)$ with $t \in \mathbb{R}$ so that for $t < -1$ $B(t) = B_{-}$ and $t > 1$ $B(t) = B_{+}$. Let A be the corresponding $4d$-connection on and

$\mathbb{R} \times P = Q \to Z = \mathbb{R} \times Y$. Then we make the following definitions.

$$\mathcal{A}(\alpha, \beta) = A + L^2_2(Z; T^*Z \otimes \mathrm{ad}P)$$

$$\mathcal{G} = \{g \in 1 + L^2_3(Z, \mathrm{End}(E)) | gg^* = 1.\}$$

Easy to check the $\mathcal{G}$ is a Hilbert manifold and the that multiplication and inversion are smooth maps. Since gA is locally $gag^{-1} + gag^{-1}$ $\mathcal{G}$ acts (smoothly) on $\mathcal{A}(\alpha, \beta)$

$$\mathcal{B}(\alpha, \beta) = \mathcal{A}(\alpha, \beta)/\mathcal{G}.$$

In fact $\mathcal{B}$ is Hilbert manifold with a local chart given about $[A]$by

$$\{[A + a] | {d_A}^* a = 0, \|a\|_{L^2_{2,A}} < \epsilon\}$$

The Coulumb slice condition.

- If stabilizer of end points is not $Z(G)$ this is not a good definition.
- The definition depends on a homotopy class $\gamma \in \pi_1(\mathcal{B}_Q; \alpha, \beta)$ so write $\mathcal{A}_\gamma(\alpha, \beta)$ or $\mathcal{B}_\gamma(\alpha, \beta)$ when necessary.

For $A \in \mathcal{A}_\gamma(\alpha, \beta)$

$$F_A^+ \in L_1^2(Z, \Lambda^+(Z) \otimes \mathrm{ad} P)$$

Set

$$M_\gamma(\alpha, \beta) = \{[A] \in \mathcal{B}(\alpha, \beta) | F_A^+ = 0\}$$

We'd like to see that $M_\gamma(\alpha, \beta)$ has a Kuranishi model locally structure and that

$$M(\alpha, \beta) = \bigcup_{\gamma \in \pi_1(\mathcal{B}_P, \alpha, \beta)} M_\gamma(\alpha, \beta)$$

Write $A = B + b + cdt$ where B is a pull back connection. Then

$$F_A = F_B + dt \wedge \dot{b} + d_B b - dt \wedge d_B c + \frac{1}{2}[b \wedge b] + dt \wedge [c \wedge b]$$

$$\begin{aligned} 0 = F_A^+ = dt \wedge (\dot{b} + *F_B + d_B c + *d_B b + *\frac{1}{2}[b \wedge b] + [c \wedge b]) \\ + *\dot{b} + F_B + *d_B c + d_B b + \frac{1}{2}[b \wedge b] + *[c \wedge b]. \end{aligned}$$

The slice condition

$$\begin{aligned} 0 = -\mathbf{d}_B^*(b + cdt) &= *_4 \mathbf{d}_B *_4 (b + cdt) \\ &= *\mathbf{d}_B dt \wedge *b + *c \\ &= *_4(-dt \wedge (d_B * b + dt \wedge *\dot{c} + d_B * c) \\ &= \dot{c} + d_B^* b + dt \wedge d_B^* c. \end{aligned}$$

Linearized equations at B.

$$0 = \begin{bmatrix} \dot{b} \\ \dot{c} \end{bmatrix} + \begin{bmatrix} *d_B & d_B \\ d_B^* & 0 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}$$

This of the form $\frac{d}{dt} + D$ where D is a first order self-adjoint elliptic operator.

Theorem

If D is an invertible first order self-adjoint elliptic operator acting on a vector bundle $F \to Y$ then

$$\frac{d}{dt} + D : L^2_k(Z; F) \to L^2_{k-1}(Z, F)$$

is invertible.

Sketch of proof in the case $k = 1$. Note that

$$\begin{aligned}
\|(\frac{d}{dt} + D)u\|^2_{L^2(Z)} \int_Z \langle (\frac{d}{dt} + D)u, (\frac{d}{dt} + D)u\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 + 2\langle \frac{du}{dt}, Du\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 + \frac{d}{dt}\langle u, Du\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 * 1 \\
&\geq C\|u\|^2_{L^2_1}.
\end{aligned}$$

Thus $\frac{d}{dt} + D : L^2_k(Z; F) \to L^2_{k-1}(Z, F)$ is injective with closed range. Exercise: Show the range is dense. Generalize this to the case of general k.

This can be used to prove in general that provided the Extend Hessian of $\mathcal{L}$

$$EH_B = \begin{bmatrix} *d_B & d_B \\ d_B^* & 0 \end{bmatrix}$$

is invertible for $B = B_\pm$ the linearized ASD equations and gauge fixing are Fredholm so the package we use to analyze the moduli space on closed manifolds carries over to this case. EH_B plays the role of the Hessian in finite dimensional (and no group action) Morse theory.

What does invertibility mean? If

$$\begin{bmatrix} b \\ c \end{bmatrix}$$

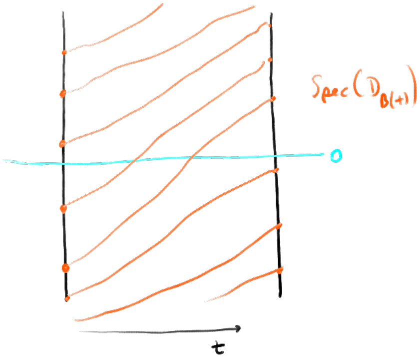
is in the kernel of EH_B and B is flat the b is a harmonic representative of $H^1(Y, \mathrm{ad}_B)$ and $c \in H^0(Y, \mathrm{ad}_B)$. In other words the B must be

- Irreducible
- Infinitesimally isolated (non-degenerate critical point.)

Given $A \in \mathcal{A}_\gamma(\alpha, \beta)$ write $A = B + cdt$ where $B = B(t)$ is path in $\mathcal{A}_Q$ and we can ask what is the index of

$$\frac{d}{dt} + D_B.$$

We now have a family of self-adjoint operators $D_{B(t)}$. Such a family has a spectral flow. Below we have a spectral flow of $+2$.



Note the self-adjoint Fredholm operators $\mathcal{SF}$ has three components

$$\mathcal{SF} = \mathcal{SF}_{-} \cup \mathcal{SF}_{o} \cup \mathcal{SF}_{+}$$

where $\mathcal{SF}_{\pm}$ are the essentially positive and negative operators. $\mathcal{SF}_{\pm}$ are contractible while

$$\Omega\mathcal{SF}_{o} \equiv \mathbb{Z} \times BO(\text{ or } \mathbb{Z} \times BU \text{ for complex ops}).$$

Theorem (Atiyah-Patodi-Singer..)

$$Ind(\frac{d}{dt} + D_t) = sf(D_t)$$

Idea of proof. Index is homotopy invariant so homotope the family so that D_t all have the same eigenvectors. Then consider

$$T_\pm = \frac{d}{dt} + \pm \tanh(t) : L^2_1(\mathbb{R}) \to L^2(\mathbb{R}).$$

The kernel of T_+ is spanned by $\mathrm{sech(t)}$ while for T_- the kernel is spanned by $\cosh(t)$. Since $T_\pm^* = -T_\mp$ when

$$\mathrm{Ind}(T_\pm) = \pm 1 = sf(\pm \tanh(t)).$$

.

Note the spectral flow is the what remains of the Morse index. $\mu(\alpha)$ should be the number of negative eigenvalue of the Hessian. Infinite in this case. However in the finite dimensional situation

$$\dim(M(x,y)) = \mu(y) - \mu(x) = sf(Hess_{\gamma(t)} f)$$

and we can make sense in this case of the difference

$$\mu(\beta) - \mu(\alpha) = sf(EH_{\gamma(t)})$$

If $\gamma : S^1 \to \mathcal{B}_Q$ is a closed loop i.e. or $\tilde{\gamma} : [0,1] \to \mathcal{A}_Q$ and so that there is a gauge transformation with $g\tilde{\gamma}(0) = \tilde{\gamma}(1)$ then *sf* may be non-zero! Indeed let A be the connection in $S^1 \times Q$ with

$$sf(D_\gamma(t)) = \mathrm{Ind}(D_A) = 8k - \frac{3}{2}(0+0) = 8k = 8\deg(g).$$

N.B. if Q is an SO_3 bundle $k \in \frac{1}{2}\mathbb{Z}$.

Lemma

Let A be a finite energy ASD connection on the half-cylinder $[0,\infty) \times Y$. Suppose that all flat connections Γ on Y have $H^1(Y, \mathrm{ad}\Gamma) = 0$. Then there is a flat connection Γ, $T > 0$ and gauge transformation g on $[T,\infty) \times Y$ so that

$$g \cdot A = \Gamma + a$$

and $a \in L^2_2([T,\infty) \times Y; T^* \otimes \mathrm{ad}P)$.

The first step in the proof is

Lemma

Let A_i be a sequence of ASD connection on a four manifold X with boundary. Suppose that $\int_X |F_{A_i}|^2 \mapsto 0$. Then there is a flat connection Γ and a subsequence still call A_i and a sequence of gauge transformations g_i so that

$$g_i \cdot A_i \mapsto \Gamma$$

in the C^∞-topology on compact subsets of the interior of X.

Proof. The assumption that the curvature has L^2 tending to zero preclude bubbling so the rest follows from the version of Uhlenbeck's theorem we have already proved.

Suppose Y is a homology S^3.

- $R(Y) = \{\rho : \pi_1(Y) \to SU_2\}/conj$ be the space of conjugacy classes of representations into SU_2.
- $R^*(Y)$

Recall that we have $R(Y) = [\mathfrak{t}] \cup R^*(Y)$ if an only Y is a $\mathbb{Z}$-homology sphere where $[\mathfrak{t}]$ is the trivial gauge equivalence class. since if $\rho : \pi_1(Y) \to SU_2$ is reducible then $\rho(\pi_1) \subset U_1 \subset SU_2$ and ρ factors through a non-trivial representation

$$\tilde{\rho} : H_1(Y, \mathbb{Z}) \to U_1.$$

Assumption. Suppose that Y is a homology sphere and for all $\alpha, \beta \in R(Y)$ and all $\gamma \in \pi_1(\mathcal{B}_Q; a, b)$

$$M_\gamma(a, b)$$

cut out transversally by the ASD equations. Note that if a or b are in $R^*(Y)$ then all $[A] \in M_\gamma(a, b)$ are irreducible and we have Then we have

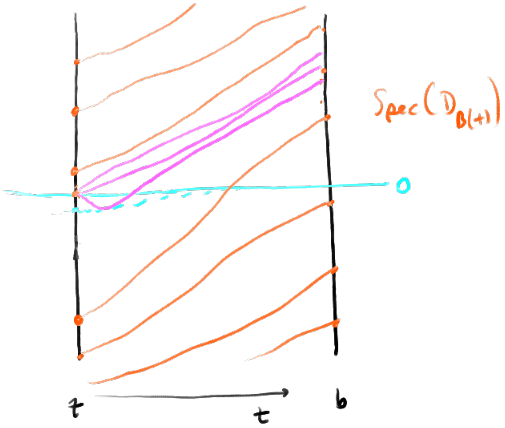
$$\dim(M_\gamma(a, b)) = sf_\gamma(a, b).$$

Let

$$\breve{M}_\gamma(a, b) = M_\gamma(a, b)/\mathbb{R}.$$

So that $\dim(\breve{M}_\gamma(a, b)) = sf_\gamma(a, b) - 1$.

What is the dimension of $M_\gamma(a, \mathfrak{t})$ or $M_\gamma(\mathfrak{t}, b)$.
Define $sf_\gamma(\mathfrak{t}, b)$ as follows



and $sf_\gamma(a, \mathfrak{t})$ similarly.

Then

$$dim(M_\gamma(a, \mathfrak{t}) = sf_\gamma(a, \mathfrak{t}) \text{ and } dim(M_\gamma(\mathfrak{t}, a) = sf_\gamma(\mathfrak{t}, b)$$

but

$$sf_{\gamma_1}(a, \mathfrak{t}) + sf_{\gamma_2}(\mathfrak{t}, b) + 3 = sf_{\gamma_1+\gamma_2}(a, b)!$$

What is the Uhlenbeck compactification of this space? An ideal instanton, $\mathcal{I}$ is a pair consisting of a point in a symmetric product of Z and an ASD connection on Z. Set

$$IM_\gamma(a,b) = \bigcup_k Sym^k(Z) \times M_{\gamma-k}(a,b).$$

for the space of Ideal instantons from a to b in the homotopy class γ. The dimension of a typical piece

$$Sym^k(Z) \times M_{\gamma-k}(a,b)$$

is

$$4k + sf_\gamma(a,b) - 8k = sf_\gamma(a,b) - 4k$$

The moduli space of ideal instantons mod translation in the homotopy class γ is

$$I\check{M}_\gamma(a,b) = IM_\gamma(a,b)/IR \ni \check{\mathcal{I}}$$

where now the typical piece has dimension

$$sf_\gamma(a,b) - 4k - 1$$

A broken path of ideal instantons from a to b in the homotopy class γ is an ordered set $(\check{\mathcal{I}}_1, \ldots, \check{\mathcal{I}}_l)$ with $\check{\mathcal{I}}_i \in \check{M}_{\gamma_i}(a_{i-1}, a_i)$ and so that

- if $a_{i-1} = a_i$ and $\gamma_i = 0$ then $k > 0$.
- $a_0 = a, a_l = b$
- $\gamma_1 + \ldots + \gamma_l = \gamma$.

Let

$$\check{M}^+_\gamma(a, b)$$

be the set of broken path of ideal instantons.

Note that the dimension of a stratum

$$(Sym^{k_0}(Z)\times M_{\gamma_0-k_0}(a_0,a_1))/\mathbb{R}\times\ldots(Sym^{k_l}(Z)\times M_{\gamma_l-k_l}(a_{l-1},b))/\mathbb{R}$$

of $\breve{M}^+_\gamma(a,b)$ is

$$sf_\gamma(a,b)-l-4(k_0+\ldots+k_l)-3t$$

where t is the number times the trivial connection appears amongst the $a_1,\ldots,a_{l-1}$.

Theorem

With the topology of convergence upto bubbling mod translation. $\breve{M}^+_\gamma(a,b)$ is compact.

NB. The gauge equivalence class of the trivial connection $\mathfrak{t}$ can be among the a_i.

The (mod 2) Floer-Morse complex is

$$C_* = \bigoplus_{a \in R^*(Y)} \mathbb{Z}_2 a$$

with

$$\partial(a) = \sum_{b \in R^*(Y)} \sum_{sf_\gamma(a,b)=1} \sharp\{[A] \in \check{M}_\gamma(a,b)\}\beta.$$

The homology of this complex is denoted

$$\mathbb{I}_*(Y).$$

Why is this $\partial^2 = 0$? The component of $\partial^2(a)$ along c is

$$\sum_{b \in R^*(Y)} \sum \sharp(\check{M}_{\gamma_1}(a,b))\sharp(\check{M}_{\gamma_2}(b,c))$$

Consider the compactification

$$\breve{M}^{+}_{\gamma}(a, c)$$

where in this case we have $sf_{\gamma}(a, c) = 2$. Only the non-negative dimensional strata appear so

$$2 - l - 4(k_1 + \ldots + k_l) - 3t \geq 0.$$

Thus $k_i = 0, t = 0$ and $l = 1, 2$. So the only ends of $\breve{M}^{+}_{\gamma}(a, c)$ are

$$\breve{M}_{\gamma_1}(a, b) \cup \breve{M}_{\gamma_2}(b, c).$$

Since these appear as ends the of a one dimensional moduli space there is an even number.

For SU_2 construction works if there are no representations with $U_1 \subset SU_2$ stabilizer.

For general N we can work with PU_N-bundles. $P \to Y$ classified by a characteristic class:

$$w(P) \in H^2(Y, \pi_1(PU_N)) = H^2(Y, \mathbb{Z}/N\mathbb{Z}).$$

More generally for PU_N the construction works if there are no representation with non-trivial stabilizer. With not trivial topology can guarantee this if there is a surface $\Sigma \hookrightarrow Y$ so that $\langle w(P), [\Sigma] \rangle$ is a unit in $\mathbb{Z}/N\mathbb{Z}$.

These theory have a grading mod $2N$.

The action of homology classes

Just as the homology of a space is acted on by cohomology via the cap product the Floer homology is acted on by the cohomology of space of gauge equivalence classes of connections. Now this has the homotopy type of

$$Maps_P(X, BSU_2).$$

and so there is an evaluation map

$$ev : X \times Maps_P(X, BPU_N) \to BPU_N.$$

Thus we define maps

$$\mu_\ell : H_*(X) \to H^{2\ell-*}(\mathcal{A}/\mathcal{G})$$

by

$$ev^*(c_\ell)/\alpha.$$

The resulting cohomology classes lead to the operators.

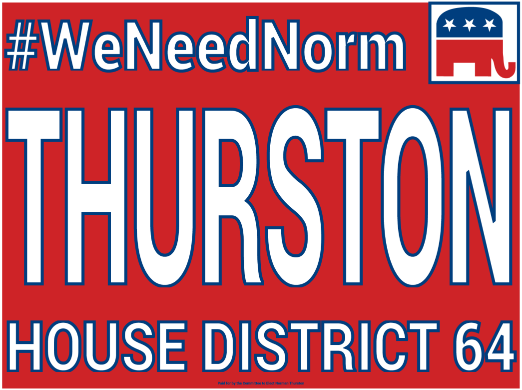
$$\Phi_X(\alpha)_\ell.$$

Defined by

$$\Phi_X(\alpha)_\ell([\rho]) = \sum_{\substack{[\sigma] \\ sf_\gamma([\rho],[\sigma])=2\ell-|\alpha|}} \langle \mu_\ell(\alpha), M_{X,w}([\rho],[\sigma])[\sigma] \rangle$$

where the sum is over moduli spaces of dimension $2\ell - |\alpha|$.

Floer homology and the Thurston Norm



The spectrum of the operators $\Phi(\alpha)_2 : \mathbb{I}(Y)_2 \to \mathbb{I}(Y)_2$ (and $\Phi(y)_2 : \mathbb{I}(Y)_2 \to \mathbb{I}(Y)_2$). determines the Thurston Norm.
If $\alpha \in H_2(Y;\mathbb{Z})$

$$Spec(\Phi(\alpha)_2) \subset \{-\|\alpha\|_T, -\|\alpha\|_T + 2, \dots, \|\alpha\|_T - 2, \|\alpha\|_T\}.$$

Furthermore $\pm\|\alpha\|_T \in Spec(\Phi(\alpha)_2)$ (old theorem of

Sutured Manifolds

Important to extend the theory to manifolds with boundary. Borrow Gabai's sutured manifold technology. A sutured three manifold is a pair (Y, γ).

- A three manifold with boundary, Y.
- A union of simple closed curves γ on $R = \partial Y$ dividing ∂Y into two components $R_{\pm}$ so that $\partial R_+ = \gamma = \partial R_-$.

Important examples. Product sutured manifold. Let Σ be a two manifold with boundary.

$$Y = [-1, 1] \times \Sigma, \gamma = 0 \times \partial\Sigma.$$

Sutured Manifold Decomposition

Given an oriented surface $S \subset Y$ with $\partial S \subset \partial Y$ and ∂S with boundary in (Y, γ) we can cut along S to get

$$(Y, \gamma) \rightsquigarrow_S (Y', \gamma')$$

Theorem

(Gabai) Any sutured three manifold (Y, γ) (irreducible) can be decomposed along successive incompersible surfaces

$$(Y, \gamma) \rightsquigarrow^{S_0} (Y_1, \gamma_1) \rightsquigarrow^{S_1} (Y_2, \gamma_2) \dots \rightsquigarrow^{S_{n-1}} (Y_n, \gamma_n)$$

where each S_i is incompressible in (Y_i, γ_i) and (Y_n, γ_n) is product sutured manifold.

Cheap and dirty sutured Floer homology.

Juhász defined a sutured Floer homology in Heegaard Floer context for a balanced suture manifold (Y, γ). Balanced means $\chi(R_+) = \chi(R_-)$.

$$SHH(Y, \gamma)$$

Given (Y, γ) pick an high genus connected surface R_0 with ∂R_0 diffeomorphic to γ. Choose $\phi : [-1, 1]R_0 \to \partial Y$ a diffeormorphism onto a tubular neighborhood of γ.

$$\tilde{Y} = Y \cup_\phi [-1, 1] \times R_0$$

$$\partial \tilde{Y} = 1 \times R_0 \cup R_+ \coprod \{-1\} \times R_0 \cup R_- = \tilde{R}_+ \coprod \tilde{R}_-.$$

Balanced implies $\chi(\tilde{R}_-) = \chi(\tilde{R}_+)$. Choose a diffeomorphism $\tau : \tilde{R}_+ \to \tilde{R}_-$. Form

$$\bar{Y} = \tilde{Y}/\tau$$

Then $\bar{R} \subset \bar{Y}$. For example. If (Y, γ) is a product sutured manifold then $\bar{Y}$ is fibered three manifold and $\bar{R}$ is a fiber, for correct τ is a product.

The Floer homology of $S^1 \times \Sigma$ is known (for $G = PSU_2$ and $w = P.D.(S^1 \times pt)$ thanks to work of Munoz. In particular $Dim(\mathbb{I}(Y)_2 | (\Phi(\alpha)_2 - \|\alpha\|_T \& \Phi(y)_2 = 4\} = 1$ as per the fibration result. Set

$$S\mathbb{I}(Y,\gamma) = (\mathrm{Ker}(\Phi(\alpha)_2 + \chi(\bar{R}))^m \cap \mathrm{Ker}(\Phi(y)_2 - 4)^m$$

at operators on $\mathbb{I}(\bar{Y})$ and for m large. The (surprising) fact is that $S\mathbb{I}(Y,\gamma)$ is independent of the choices made. (Kronheimer-M).

Behavior of $S\mathbb{I}(Y,\gamma)$ under sutured manifold decomposition

The basic result Juhász proves for $SHH(Y,\gamma)$ carries over to $S\mathbb{I}(Y,\gamma)$.

Theorem

If $S_i \subset Y_i$ and $(Y_i,\gamma_i) \overset{\rightsquigarrow}{S_i} (Y_{i+1},\gamma_{i+1})$ as in Gabai's theorem then we have $S\mathbb{I}(Y_{i+1},\gamma_{i+1})$ is a direct summand of $S\mathbb{I}(Y_i,\gamma_i)$

Getting a knot invariant.

$K \subset S^3$, let $Y = S^3 \setminus N(K)$ and let γ a pair of oppositely oriented meridians. Define

$$H\mathbb{I}(K) = S\mathbb{I}(Y, \gamma).$$

In fact for suitable choice $\bar{Y} = S^3 \setminus N(K) \cup S^1 \times (T^2 \setminus D^2)$. In this model a Seifert surface of genus g gets completed to surface, $\hat{S}$ of genus $g + 1$. Unless $g = 0$ there at least eigenspace of $\mu(\hat{S})$ that are non zero thus:

$$rank(H\mathbb{I}(K)) = 1$$

if and only $K = U$.

Brief introduction to Khovanov homology.

- For an unlink $\mathbb{U}_N$ of N components

$$Kh(\mathbb{U}_N) \sim A^{\otimes N}$$

  where A is $H_*(S^2)$
- The cobordisms $\mathbb{P} : \mathbb{U}_2 \to \mathbb{U}_1$ and copants $\bar{\mathbb{P}} : \mathbb{U}_1 \to \mathbb{U}_2$ induce the two natural maps:
  Intersection products

$$Kh(\mathbb{U}_2) \sim A \otimes A \xrightarrow{\mathbb{P}_*} Kh(\mathbb{U}_1) \sim A,$$

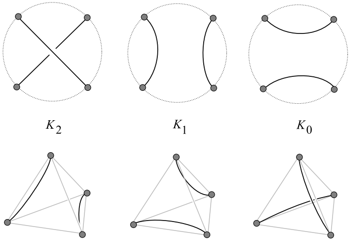
  and diagonal inclusion

$$\Delta : A \to A \otimes A.$$

  So that $X = [x]$ and $1 = [S^2]$ and

  $\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1, \Delta(X) = X \otimes X.$

- The Skein Exact Sequence. Given a knot diagram form the "Cube of resolutions".
  Consider three knots, K_2, K_1, K_0 which identical in the complement of a ball and in the ball looks like:



Then there is an exact triangle:

$$Kh(K_0) \to Kh(K_1) \to Kh(K_2) \to Kh(K_0).$$

where the maps are those induced by the (local) cobordisms.

- "Universality"

Some examples and properties.

$Kh(\text{unlink})$	A
$Kh(\text{N-component unlink})$	$A^{\otimes N}$
$Kh(\text{Trefoil})$	$\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$

Some Coincidences

Consider the space of representations (N.B not conjugacy classes)

$$R_I(Y,K) = \{\rho : \pi_1(Y \setminus K)) \to U_2 | spec\rho(\mu) = \{\pm 1\}\}$$

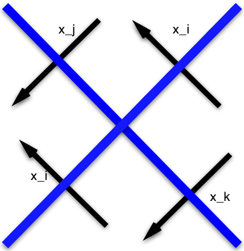
Thus $\rho(\mu)^2 = 1$ and $det(\rho(\mu)) = -1$. The set of such elements is a 2-sphere.
For an unlink of k-components we get $\times_k S_I^2$.

For knots in S^3 it is somewhat easier to a get a description of this representation space. Consider the Wirtinger presentation for the fundamental group of the knot complement.

$$\pi_1(S^3 \setminus K) = \langle x_1, x_2, \ldots, x_n | r_1, \ldots r_{n-1} \rangle$$

where each r_s are is of the form

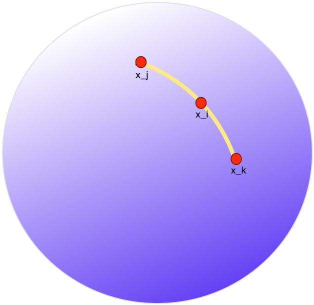
$$x_i x_j x_i^{-1} = x_k.$$



Now $p \in S^2$ acts on S^2 by conjugation as the symmetric space involution $\tau_p : S^2 \to S^2$ of reflection along geodesics through p. Thus since the x_i's are meridians we see that the three points

$$\rho(x_j), \rho(x_i), \rho(x_k)$$

lie on the same geodesic with $\rho(x_i)$ being (a) midpoint between the other two.



For the trefoil the presentation is simply $\langle x, y, z | xyx^{-1} = z \text{ and cyclic}\rangle$. Thus we are looking for triples of points on S^2 lying on the same geodesic so that each point a midpoint of the other two. There are two possibilities, either all three points are the same or they 3 equally spaced labelled points on a geodesic.

$$R_I(S^3, T_{2,3}) = S^2_I \coprod \mathbb{RP}^3$$

For a $(2, p)$-torus knot the story is similar. The presentation is

$$\langle x_1, x_2, \ldots, x_p | x_{i+1} x_i x_{i+1}^{-1} = x_{i+2} \rangle.$$

Thus the repsentation space is p ordered points lying on the same geodesic so that x_{i+1} is the midpoint of x_i and x_{i+2}. There possibilities

- x_1 and x_2 (hence all the points are the same).
- x_1 and x_2 are antipodal in which case p is even and all odd points are x_1 and all even points are x_2.
- The angle of x_1 and x_2 is $2\pi k/p$ where $0 < k < p/2$.

$$R_I(S^3, T_{2,2m+1}) = S_I^2 \coprod_m \mathbb{RP}^3$$

$$R_I(S^3, T_{2,2m+2}) = S_I^2 \coprod S_I^2 \coprod_m \mathbb{RP}^3$$

Taking the homology in any of these cases gives the Khovanov homology (without a grading), of the corresponding knot. Of course to get in invariant with good properties simply taking the homology is too naive (but we all know what to do), should use instanton Floer homology.

Instanton Floer Homology (on Orbifolds)

To get knots in the picture we generalize this story to orbifolds. The construction generalizes in a rather straightforward manner to $\mathbb{Z}_2$-orbifolds. Consider $P \to \hat{Y}$ an orbifold principal U_2 bundle with orbifold locus a link $K \subset Y$. The determinant δ_P is an orbifold line bunlde. The orbifold fundamental group

$$\pi_{1,orb}(\hat{Y}) = \pi_1(Y \setminus K)/\langle \mu^2 = 1 \rangle.$$

Hence the representation space is a (union of) components of projective representation of $\pi_{1,orb}(\hat{Y})$.
We will need to fix a connection in $det(P) = \delta$. We'll keep track of this by choosing a one manifold $\omega \subset Y$ with $\partial\omega \subset K$ and require the connection the connection in δ to be flat on $Y \setminus (\omega \cup K)$ have holomony -1 on any loop linking $\omega \cup K$ once.

Critical points of CS mod gauge are now conjugacy classes of representations $\rho : \pi_1(Y \setminus (K \cup \omega)) \to U_2$ s.t.

- $det(\rho(\gamma)) = -1$ if γ links $\omega \cup K$.
- $\rho(\mu)^2 = 1$ if μ links K once.

Note that neither of these conditions uses an orientation of the knot. Actually a little more care is require to say this carefully. More on this later.

The Morse homology of CS on connection on $P \to \hat{Y}$ is denoted

$$\mathbb{I}_*(Y, K, \omega).$$

This a $\mathbb{Z}_4$ graded $\mathbb{Z}$-module.
We can do this provided the critical points of CS are away from the singular points of the quotient space $\mathcal{A}/\mathcal{G}$, in other words there are no reducible representations.

Avoiding Reducible Representations

We need to avoid reducible representations. We can achieve this by adding a Hopf link in a little ball in our three manifold and where the part of ω contained in that ball is a curve joining the two components. There is a unique up to conjugacy representation of the Hopf Link complement where the generator are sent to matrices X and Y satisfying

$$X^2 = 1, \; Y^2 = 1, \; \mathrm{and} \det(\mathrm{XYX^{-1}Y^{-1}}) = -1$$

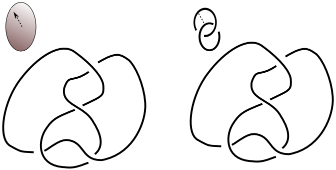
One can check that:

$$\begin{aligned} X &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ Y &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned} \tag{1}$$

solve this, uniquely upto conjugacy.

Getting knot invariants.

Given link $K \subset Y^3$ as indicated before we can naturally from and new link to which the theory can be applied.



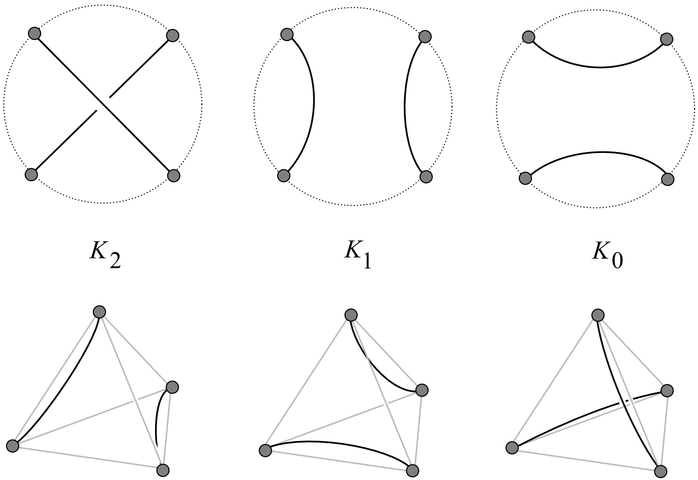
$(Y, K) \qquad (Y, K^\sharp, \omega)$

And define $I^\sharp(Y, K) = I(Y, K^\sharp, \omega)$, if $Y = S^3$ sometimes it gets dropped from the notation.

This finally gives the correct representation variety!.

$$Hom(\pi_1(Y \setminus K), SU_2) = Hom(\pi_1(Y \setminus K^\sharp, SU_2)/conj.$$

Exact sequences and spectral sequences Consider three knots, K_2, K_1, K_0 which identical in the complement of a ball and in the ball looks like:



Such a ball is called a crossing. An orientation of K gives a crossing a sign.

These knots are cobordant via surfaces $S_{2,1}, S_{1,0}, S_{0,2}$

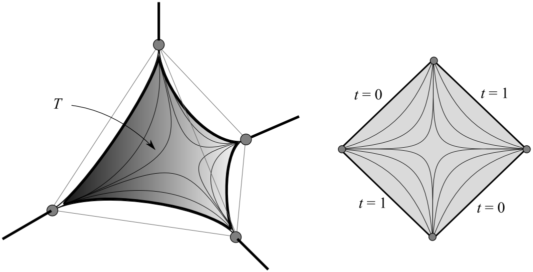


Figure: The twisted rectangle T that gives rise to the cobordism $S_{2,1}$ from K_2 to K_1.

These cobordisms induce maps

$$\ldots \to I^\sharp(K_2) \stackrel{\Phi_{2,1}}{\to} I^\sharp(K_1) \stackrel{\Phi_{1,0}}{\to} I^\sharp(K_0) \stackrel{\Phi_{0,2}}{\to} \ldots .$$

In fact this is an exact triangle.

This follows from observing that for example

$$Z_0 = (I \times S^3, S_{2,1}) \circ (I \times S^3, S_{1,0})$$

is diffeomorphic to

$$Z_1 = (I \times S^3, S_{2,0}) \# (S^4, \mathbb{RP}^2_+).$$

where $S_{2,0}$ is the cobordism $S_{0,2}$ viewed backwards.

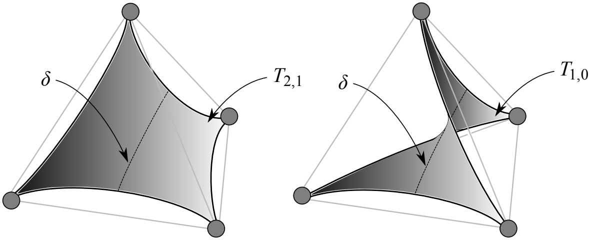


Figure: Curve δ has a neighborhood which is a möbius band

This means there is a one parameter family $g_t, t \in [0,1]$ of metrics on Z. The two ends points of which realize the two decompositions above. For simple geometric reasons the moduli space for the metric g_1 is empty.
This means we can define a chain homotopy:

$$H_{0,2}(\alpha) = \sum_{\beta} \#(\cup_t M_{g_t}(\alpha, \beta))\beta$$

which verifies

$$\Phi_{1,0} \circ \Phi_{2,1} = \partial_0 H_{2,0} + H_{2,0}\partial_2.$$

These chain homotopies allow us to define a chain maps

$$C_0 \leftrightarrow Cone(\Phi_{2,1})$$

given by $(\Phi_{0,2}, H_{0,1})$ and $(H_{2,0}, \Phi_{1,0})$. These maps are chain homotopy inverses proving the existence of the long exact sequence.

To see that when we stretch out the $(S^4, \mathbb{RP}^2_+)$ pair the moduli space eventually becomes empty note that on $(S^4, \mathbb{RP}^2_+)$ (this one is double branch covered by $-\mathbb{C}P^2$) we saw there was as unique irreducible solution. (S^3, U) there is a unique flat connection and it is reducible with stabilizer U_1. Thus when you stretch the moduli space becomes empty.

The distguished triangle detection lemma. (Ozsvath and Szabo.) cf Seidel

Suppose we have:

- a sequence of complexes $\{(C_i, d_i)\}_{i\in\mathbb{Z}}$,
- a sequence of anti-chain maps $f_i : C_i \to C_{i-1}$,
- a sequence $j_i : C_i \to C_{i-2}$ chain homotopies

  $$d_{i-2}j_i + j_i d_i + f_{i-1}f_i = 0$$

  for all $i \in \mathbb{Z}$,

so that the (chain) map:

$$j_{i-1}f_i + f_{i-2}j_i : C_i \to C_{i-3}$$

is chain homotopic to an isomorphism.
Then the induced maps in homology,

$$(f_i)_* : H_*(C_i, d_i) \to H_*(C_{i-1}, d_{i-1})$$

form an exact sequence; in particular for each i the anti-chain map

$$\Phi : s \mapsto (f_i s, j_i s)$$

$$\Phi : C_i \to \mathrm{Cone}(f_{i-1})$$

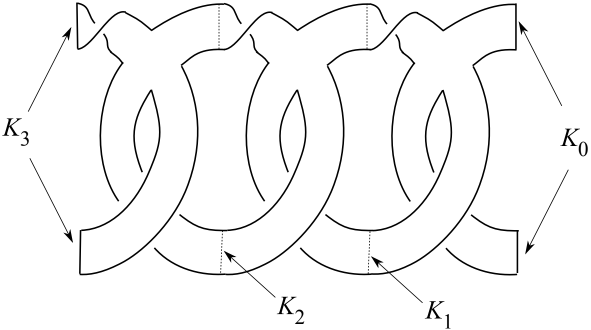


Figure: Two intersecting Möbius bands, M_{31} and M_{20} inside S_{30}.

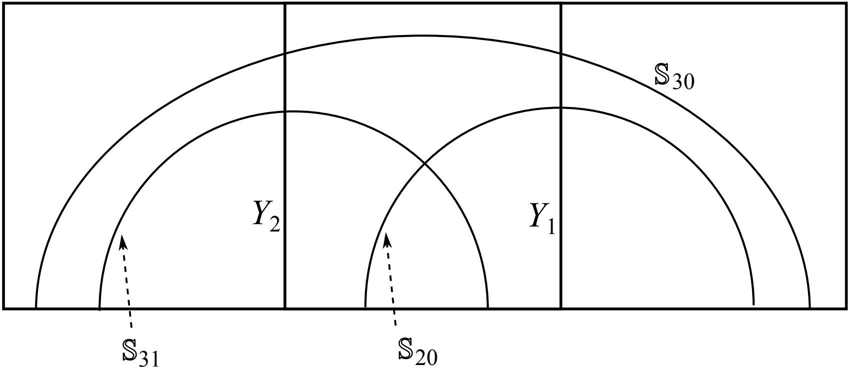


Figure: The five 3-manifolds, Y_2, Y_1, $\mathbb{S}_{30}$, $\mathbb{S}_{31}$ and $\mathbb{S}_{20}$ in the composite cobordism $([0,3] \times Y, S_{30})$.

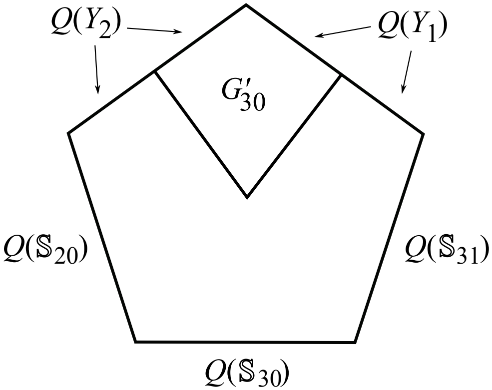
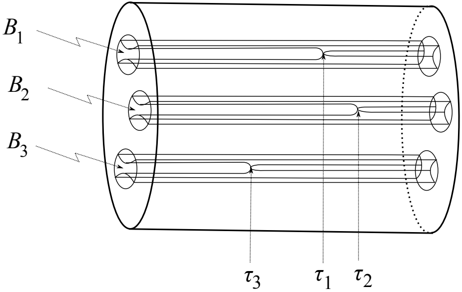


Figure: The family of metrics G_{30} containing the image of family G'_{30}.

Given a collection of crossings, $\mathbb{X} = \{B_1, \dots B_n\}$ we can construct from $K = K_{2\dots 2}$, we obtain a collection of knots, K_v where $v \in \mathbb{Z}^{\mathbb{X}}$. This family is 3-periodic in each component. We also have surfaces S_{vu} which give cobordisms from K_v to K_u. When $v = u + e_i$ such surfaces induce maps just as before. More generally when $v > u$ the surfaces come in families.



The family of metrics G_{vu} parametrized by $\tau \in \mathbb{R}^l$ ($\mathbb{R}^3$ in this example).

Specializing to the case $u, v \in \{0,1\}^{\mathbb{X}}$ note that the family of metrics admits an action of translation by $\mathbb{R}$. Let $\breve{G}_{uv}$ denote the quotient by this action. We can define maps

$$F_{vu} : C_v \to C_u$$

by the formula

$$F_{vu}(\alpha) = \sum_{\beta} \#(M^{\breve{G}_{vu}})(\alpha, \beta))\beta$$

where $v > u$, $F_{uu} = \partial_u$ and $F_{vu} = 0$ when $v < u$. We get a Khovanov like cube of resolutions.

$$(\mathbf{C}_{\mathbb{X}}, \mathbf{F}_{\mathbb{X}}) = (\oplus_{\mathbf{u}} \mathbf{C}_, [\mathbf{F}_{\mathbf{vu}}])$$

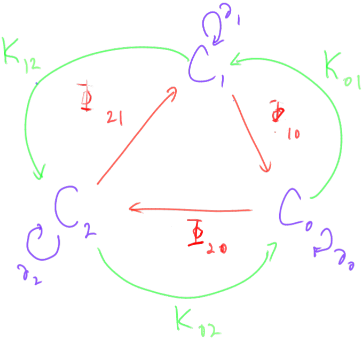
This is a chain complex with differential **F** generalizing the mapping cone of the case (N=1). It is chain homotopy equivalent with $(C(K), \partial_K)$.

If all of the resolutions K_u of $\mathbb{X}$ are unlinks we call $\mathbb{X}$ a pseudo-diagram. In this case $I^\natural(K_v)$ is is isomorphic to

$$H_*(\times_{b_0(K_v)} S^2) = \times_{b_0(K_v)} H_*(S^2).$$

If in addition we have a diagram then the complex $(\mathbf{C}_\mathbb{X}, \mathbf{F}_\mathbb{X})$ is filtered by $|v|$ and the E^1 page can be determined together with the differential. Then all the surfaces S_{vu} with $|v - u| = 1$ are pairs of pants and the E^1 page of the spectral sequence is the Khovanov complex.
A similar construction was done by Ozsvath and Szabo 2006. They related the Heegaard Floer homology of the double branched over of a link and gave a mod 2 relation between Khovanov homology and earlier and was a motivation for this work. Jonathan Bloom established a similar story for the Seiberg-Witten equations.

The distguished triangle detection lemma. (Ozsvath and Szabo.)



Φ_{ij} - anti-chain maps K_{ij} - chain homotopies

$\Phi_{10}\Phi_{21} \simeq \partial_2 K_{12} + K_{12}\partial_1$ + cyc

If $\Phi_{10}K_{01} + K_{02}\Phi_{20}$ iso

$$\Rightarrow \quad C_0 \longrightarrow \mathrm{Cone}(F_{21})$$
$$x \longmapsto (F_{20}(x), K_{01}(x))$$

is a iso.

Theorem

Kronheimer-M 2009 For links $L \subset S^3$ (with a distinguished base point) there is a spectral sequence starting with reduced Khovanov cohomology and abutting to $I^\natural(L)$.

Theorem

Kronheimer-M If K is a knot $I^\natural(K) \equiv \mathbb{Z}$ iff $K \equiv U$.

The later theorem comes from slightly earlier work carrying over Juhász' sutured Floer homology to the instanton context.

Corollary

Reduced Khovanov homology detects the unknot.

Corollary

The Rasmussen s invariant bounds the slice genus in homotopy balls.

Bouyed this success we'd like to extend this story to SU_N gauge theory and hopefully relate it to Khovanov-Rozansky homology.

We can extend this game to U_N. Consider elements $g \in U_N$ so that

$$g^2 = 1$$

Thus g has two eigenvalues ± 1 so the conjugacy classes divide up according to the dimension of the -1 eigenspace. Suppose it is N_1-dimensional. These a form a copy of the Grassmanian $Gr_{N_1,N}$ as a distinguished subset $Gr_{N_1,N} \subset U_N$.

When $N_1 = 1$ then the representation spaces for the simple examples are

- $\mathbb{CP}^{N-1}$ for the unknot or $\times_k \mathbb{CP}^{N-1}$ for unknot of k-components.
- $\mathbb{CP}^{N-1} \coprod ST\mathbb{CP}^{N-1}$ for the trefoil.

Now the homology of these spaces reproduces the Khovanov-Rozansky homology of this examples.

Repeat setup of gauge theory on orbifolds.

Given (X, S) a of a four-manifold and a smoothly embedded submanifold. Consider this pair as defining an orbifold with cone-angle π along S. When $N > 2$, S must be orientable. To see this you need to work on the level of connections. The connections one forms

$$diag(i, 0, \ldots, 0)d\theta \text{ and } diag(-i, 0, \ldots, 0)d\theta$$

are only gauge equivalent for $N = 2$ by a determinant 1 gauge transformation. We'll do gauge theory on orbifold U_N-bundles

on $P \to (X, S)$ so that along S the $\mathbb{Z}_2$ action is non-trivial. In a little while we will allow S to be a kind of singular surface called a foam. We restrict our orbifold bundles so that above a fixed point of the local $\mathbb{Z}_2$ action the bundle automorphism is one of g's above and stick for the moment to the $\mathbb{CP}^{N-1}$ case.

Again U_N connections can have complicated stabilizers so we restrict our gauge group to those that lift to SU_N.

We fix the a connection in the determinant of the U_N-bundles by keeping track of w via $\omega \subset X$ surface with $\partial\omega \subset S$

N.B. An orbifold bundle $P \to \hat{X}$ does not necessarily give rise to a bundle on X, only locally.

We'll look at moduli spaces $M_\omega(X, S)$ of orbifold connections A on orbifold bundle $P \to X$ with w represented by ω solving the ASD equations:

$$F_A^+ = 0.$$

The action and dimension formula.

As usual we set

$$\kappa = \frac{1}{4N\pi^2} \int_X tr(F_A \wedge F_A)$$

For A ASD we have $\kappa \geq 0$.

$$\kappa = \frac{1}{8N} S \cdot S \pmod{1}/2.$$

Then

$$M_\omega = \bigcup_\kappa M_{\kappa,\omega}$$

The expected dimension of $M_{\kappa,\omega}$ is

$$4N\kappa + \frac{N^2-1}{2}(\chi(X) + \sigma(X)) + (N-1)\chi(S) + \frac{N-1}{2} S \cdot S.$$

Instanton Floer Homology

With this orbifold situation in mind we consider the case of $X = \mathbb{R} \times Y$ and $S = \mathbb{R} \times K$ for some link $K \subset Y$. Consider $P \to \hat{Y}$ an orbifold principal PU_n bundle. Keep track of $w(P)$ by a choosing a one manifold $\omega \subset Y$ with $\partial\omega \subset K$.

ASD equation on $\mathbb{R} \times Y$ corresponds to the downward gradient flow for Chern-Simons function.

Critical points of CS mod gauge are now conjugacy classes of representations $\rho : \pi_1(Y \setminus (K \cup \omega)) \to SU_N$ s.t.

- $\det \rho(\gamma) = -1$ if γ links $\omega \cup K$ once
- $\rho(\gamma) \in G_1$ if γ links K once.

Avoiding Reducible Representations

We need to avoid reducible representations. We need move to a more general situation where now we also allow the holonomy to be satisfy $g^N = e^{2\pi i/N}$. Enough if there is $\Sigma \subset Y$ so that all reps of $\pi_1(\Sigma \setminus \Sigma \cap (K \cup \omega))$ with given conditions are irreducible. We will avoid this issue by assume that there is a Hopf link with w a curve joining the two components with multiplicity 1.

$$XYX^{-1}Y^{-1} = e^{2\pi i/N}$$

$$\begin{aligned}
X &= \epsilon \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & 0 & \zeta^2 & \cdots & 0 \\ & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \zeta^{N-1} \end{bmatrix} \\
Y &= \epsilon \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned} \tag{2}$$

Here, $\zeta = e^{2\pi i/N}$, and ϵ is 1 if N is odd and an Nth root of -1 if N is even.

As before we construct the Morse homology of CS on $\mathcal{A}/\mathcal{G}$. It is denoted

$$\mathbb{I}_*(Y, K, \omega).$$

This now $\mathbb{Z}_{2N}$ graded $\mathbb{Z}$-module.
It is functorial under cobordism (of pairs). If $\partial(X, \Sigma, \omega) = (-Y_0, -K_0, -\omega_0) \coprod (Y_1, K_1, \omega_1)$ then there is a map

$$\Phi_*(X, \Sigma, \omega) : \mathbb{I}_*(Y_0, K_0, \omega_0) \to \mathbb{I}_*(Y_1, K_1, \omega_1)$$

As before the cohomology ring of $\mathcal{A}/\mathcal{G}$ acts. Now

$$\mathcal{A}/\mathcal{G} \equiv Map_P((Y,K),(BSU_N, BS(U_k \times U_{N-k}))$$

So corresponding to points in Y there are maps

$$\nu_i : \mathbb{I}_*(Y,\omega) \to \mathbb{I}_{*-4}(Y,\omega),\ i = 2,\ldots N$$

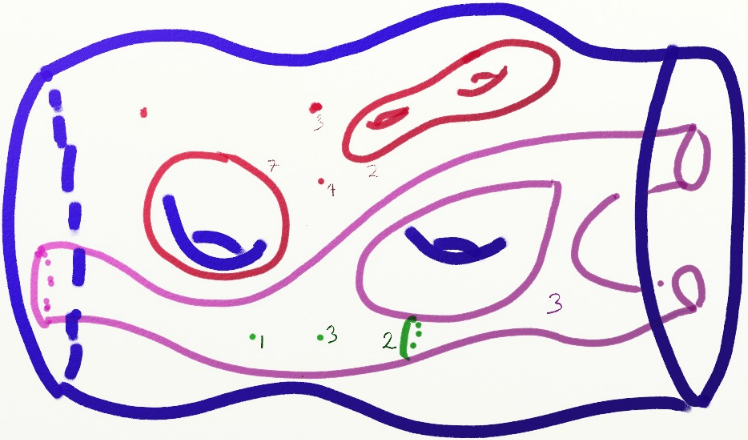
a point p on K gives rise to

$$\rho_j(p) : \mathbb{I}_*(Y,\omega) \to \mathbb{I}_{*-2}(Y,\omega),\ j = 1,\ldots,k$$

Finally a surface Σ in Y (disjoint from K) gives rise to a operators

$$\mu_i([\Sigma]) : \mathbb{I}_*(Y,\omega) \to \mathbb{I}_{*-2}(Y,\omega),\ i = 2, N$$

These are again commuting operators.



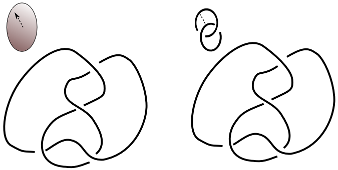
3
7
2
1
3
1
3
2

A typical cobordism of pairs decorated with 0 and 1 cycles on the two manifold and $0, 1, 2, 3$-cycles on the four manifold.

- Each component of the 2-manifold is labelled by an integer $0 \leq N_1 \leq N$.
- Each i-cycle on the two manifold is label by an integer $1, \ldots N_1$.
- Each j-cycle on the four-manifold is labelled by an integer $2, \ldots, N$.

Getting knot invariants.

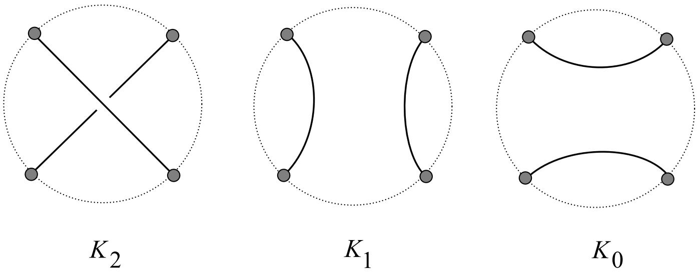
Given link $K \subset Y^3$ as indicated before we can naturally from and new link to which the theory can be applied.

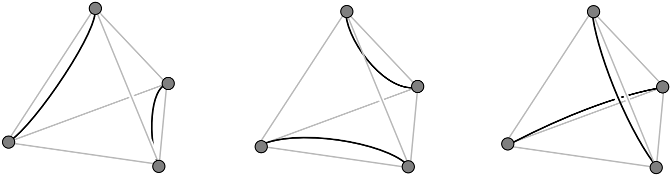


(Y, K) $\qquad$ $(Y, K^\sharp, \omega)$

And define $I^\sharp(Y, K) = I(Y, K^\sharp, \omega)$, if $Y = S^3$ sometimes it gets dropped from the notation.

Exact sequences and spectral sequences Consider three knots, K_2, K_1, K_0 which identical in the complement of a ball and in the ball looks like:





Such a ball is called a crossing. An orientation of K gives a crossing a sign.

These knots are cobordant via surfaces $S_{2,1}, S_{1,0}, S_{0,2}$

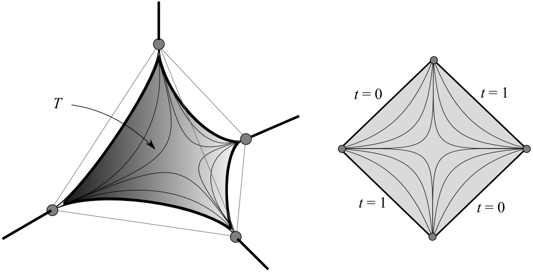


Figure: The twisted rectangle T that gives rise to the cobordism $S_{2,1}$ from K_2 to K_1.

These cobordism are not in general orientable so unless $N = 2k$, they don't induce maps. If they do

$$\ldots \to I^{\sharp}(K_2) \stackrel{\Phi_{2,1}}{\to} I^{\sharp}(K_1) \stackrel{\Phi_{1,0}}{\to} I^{\sharp}(K_0) \stackrel{\Phi_{0,2}}{\to} \ldots .$$

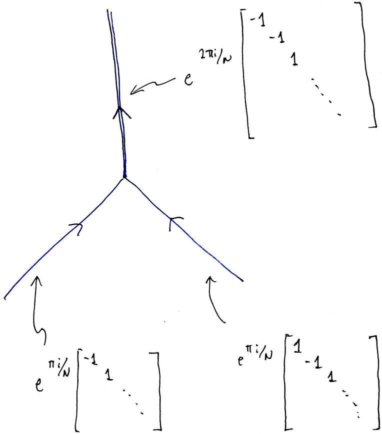
In fact this is an exact triangle when $N = 2$ (and $k = 1$) but not otherwise ($\binom{2k}{k}^2 = 2\binom{2k}{k}$ only when $k = 1$.)

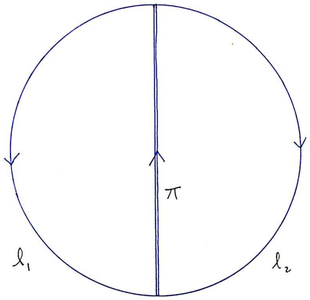
Extending the definition to webs and foams.

We are already working in the context of orbifolds so lets try a slightly more general context. The Klein 4-group $V = \mathbb{Z}_2^2$ acts on $\mathbb{R}^3$ by orientation preserving isometries. The quotient is homeomorphic to $\mathbb{R}^3$. The singular locus is the quotient of the coordinate axis, a $Y \subset \mathbb{R}^3/V$. We can consider orbifolds that a locally modeled on the open subsets of $\mathbb{R}^3/K$. Thus we can consider $K \subset Y$ to a trivalent graph and consider Y as an orbifold. To build an orbifold bundle over (Y, K) we need each edge of K to be oriented. We also associate to each oriented edge e an element of g_e of some G_k (so that $g_{-e} = g_e^{-1}$). At each vertex v with oriented edges e_1, e_2, e_3 we require

$$g_{e_1} g_{e_2} g_{e_3} = 1$$

if all the edges point into the vertex. Now there are many possibilities.


$e^{2\pi i/N}\begin{bmatrix}-1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots\end{bmatrix}$
$e^{\pi i/N}\begin{bmatrix}-1 & & \\ & 1 & \\ & & \ddots\end{bmatrix}$
$e^{\pi i/N}\begin{bmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots\end{bmatrix}$



$\pi = \ell_1 + \ell_2$

The representation space of the web is two step Flag manifold.

$$Fl_{1,2}(\mathbb{C}^N) = \{\ell, \pi | \ell \subset \pi \subset \mathbb{C}^N\}$$

Its homology agree in rank with the Khovanov-Rozansky answer.

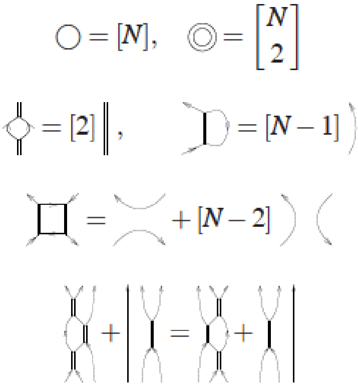
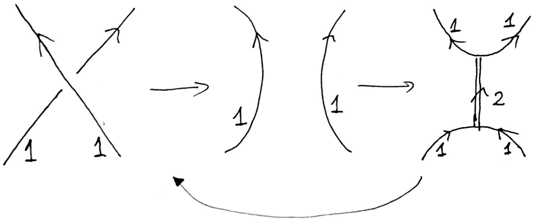


FIGURE 3. MOY web moves

from Maackay-Stosic-Vaz

The 4-dimensional pictures that naturally fit into the context of orbifolds that are locally modeled on the quotient $\mathbb{R}^4/V_8$ where V_8 is $\mathbb{Z}_2^3$ acting in an orientation preserving manner by reflecting coordinate axis. Again the quotient is $\mathbb{R}^4/V_8$ is homemorphic to $\mathbb{R}^4$. This singular locus is the image of the coordinate 2-planes and is homeomorphic to a cone on the 1-skeleton of a tetrahedron. We've checked that the tools we used before to prove the exact triangle in $N = 2$ work to prove the indicated exact triangle for webs when the edges are labelled only with 1's and 2's.
Lobb and Zenter study these representation spaces for planar webs. Ethan Street is developing the extension of this theory to tangles.

We can now revisit the skein sequence.



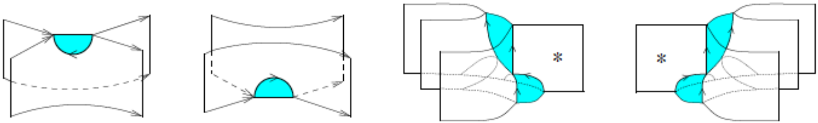


FIGURE 4. Some elementary foams

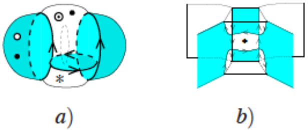


FIGURE 5. a) A pre-foam b) An open pre-foam

From Maackay-Stosic-Vaz.

We have the surfaces as before, but now orientability is not a problem. We can insert extra pieces that fix orientations.