The mean curvature flow for a convex hypersurface

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1. The mean curvature flow

The mean curvature flow

$(N, \langle\ ,\ \rangle)$: a complete Riemannian manifold

M : a compact manifold

$f_t : M \hookrightarrow N \ (0 \leq t < T)$: a C^∞-family of immersions of M into N

$$F : M \times [0, T) \to N$$

$$\underset{\text{def}}{\Longleftrightarrow} F(x, t) := f_t(x) \ \ ((x, t) \in M \times [0, T))$$

When f_t is an embedding, we set $M_t := f_t(M)$.

The mean curvature flow

H_t : the mean curvature vector of f_t

H : the section of F^*TN defined by

$$H_{(x,t)} := (H_t)_x \quad ((x,t) \in M \times [0,T))$$

Definition

$f_t \ (0 \leq t < T)$: a mean curvature flow

$$\underset{\text{def}}{\Longleftrightarrow} \frac{\partial F}{\partial t} = H \qquad \text{(MCFE)}$$

We may write (MCFE) as

$$\frac{\partial f_t}{\partial t} = H_t.$$

The mean curvature flow

We consider the case of $N = \mathbb{R}^m$ (Euclidean space).

Then H_t is described as

$$H_t = \triangle_t f_t,$$

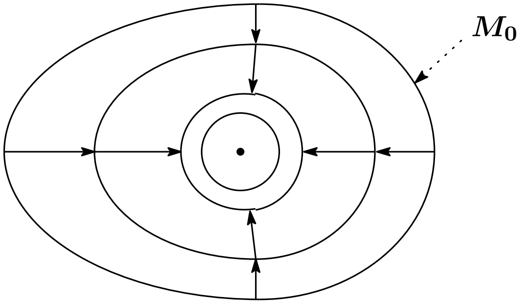
where $\triangle_t$ is the Laplacian op. of $g_t := f_t^* \langle\ ,\ \rangle$.

Hence (MCFE) is described as

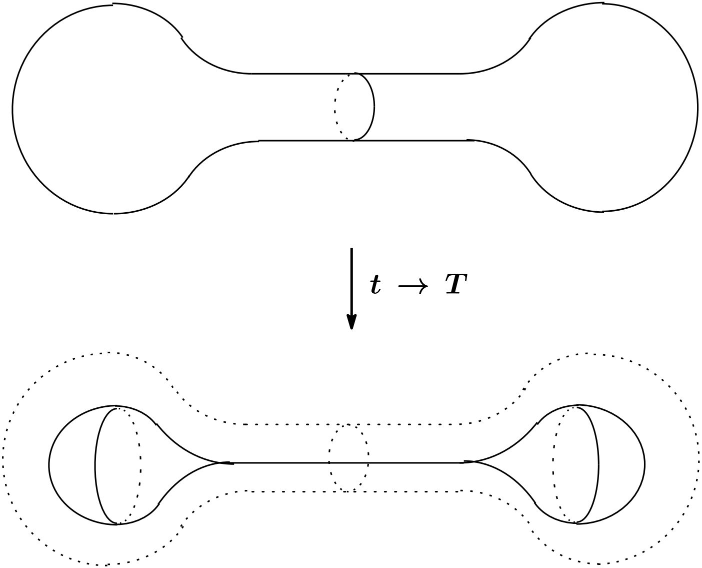
$$\frac{\partial f_t}{\partial t} = \triangle_t f_t,$$

which is a quasi-linear parabolic equation and hence the existenceness and the uniqueness of the solution of (MCFE) in short time for any smooth initial data is assured by the Hamilton's theorem.

The mean curvature flow



The mean curvature flow



The mean curvature flow

$f_t : M \hookrightarrow \mathbb{R}^m \ (0 \leq t < T)$: a mean curvature flow

Definition(self-similar solution)

If $f_t = \rho(t) f_0 \ (0 \leq t < T)$ for some positive-valued C^∞-function $\rho : [0, T) \to \mathbb{R}$, then $f_t \ (0 \leq t < T)$ is called a self-similar solution. Also, f_0 is called a self-similar immersion.

Fact

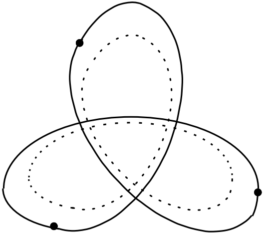
If $f_t \ (0 \leq t < T)$ is a self-similar solution, then $f_t(x) = \sqrt{2(T-t)} f_0(x) \quad (x \in M)$ holds.

The mean curvature flow

Fact

The only self-similar solutions in $\mathbb{R}^2$ are Abresch-Langer curves $\gamma_{m,n}$'s, where m is the periodic number and n is the rotational number.

The mean curvature flow



$\gamma_{3,2}$

The mean curvature flow

Fact

The mean curvature flow for an isoparametric hypersurface in a Euclidean space is a self-similar solution.

The mean curvature flow

$f_t : M \hookrightarrow \mathbb{R}^m \ (0 \leq t < T)$: a mean curvature flow

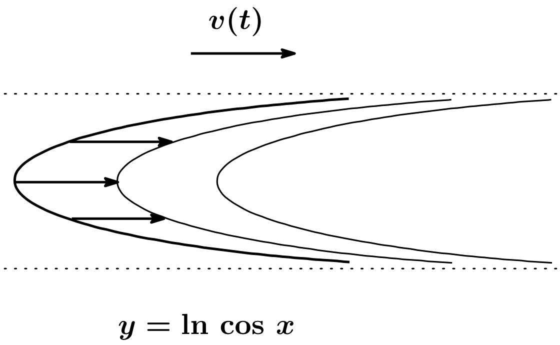
Definition(translating soliton)

If $f_t(M) = f_0(M) + v(t) \ (0 \leq t < T)$ for some constant vector $v(t)$, then $f_t \ (0 \leq t < T)$ is called a **translating soliton**. Also, f_0 is called a **translating immersion**.

Fact

The only translating soliton in $\mathbb{R}^2$ is a grim reaper.

The mean curvature flow



2. Hamilton's theorem

Hamilton's theorem

M : an n-dimensional compact manifold

V : a vector bundle over M

$\Gamma(V)$: the space of all sections of V

E : a diff. op. of order two of V

DE_f : the linearization of E at $f(\in \Gamma(V))$

$\sigma(DE_f)$: the symbol of DE_f

$f_t\ (0 \leq t < T)$: a C^∞-curve in $\Gamma(V)$

$$F : M \times [0, T) \to V$$

$$\underset{\text{def}}{\Longleftrightarrow} F(x,t) := f_t(x)\ \ ((x,t) \in M \times [0,T))$$

Hamilton's theorem

$$(*) \qquad \frac{\partial f_t}{\partial t} = E(f_t)$$

If all the eigenvalues of $\sigma(DE_f)(v)$ have the positive real parts for any $f \in \Gamma(V)$ any $v(\neq 0) \in \mathbb{R}^n$, then the PDE $(*)$ is said to be parabolic.

Hamilton's theorem

Assume that E admits a map

$$L : U \times \Gamma(V) \to \Gamma(W)$$

(U : an open subset of $\Gamma(V)$, W : a vector bundle over M)

satisfying the following conditions:

- $L(f, \cdot)$ is a diff. op. of order one for any $f \in U$
- $Q : f \mapsto L(f, E(f))$ $(f \in U)$ is a diff. op. of order one
- for any $f \in \Gamma(V)$ and any $v(\neq 0) \in \mathbb{R}^n$,
  all the eigenvalues of $\sigma(DE_f)(v)|_{N(\sigma(L(f))(v))}$
  has positive real part

$$\left(N(\cdot) \ : \text{ the nullity space of } (\cdot) \right)$$

Hamilton's theorem

Then the PDE

$$\frac{\partial f_t}{\partial t} = E(f_t)$$

is said to be weakly parabolic.

Hamilton's theorem

Theorem 2.1(Hamilton).

For any $\phi \in \Gamma(V)$, the solution of the weakly parabolic equation

$$\frac{\partial f_t}{\partial t} = E(f_t)$$

with the initial condition $f_0 = \phi$ uniquely exists in short time.

Hamilton's theorem

A immersion $f : M \hookrightarrow \mathbb{R}^m$ is regarded as a section of the trivial bundle $M \times \mathbb{R}^m$ over M.
Then (MCFE) is regarded as a parabolic equation.
Hence the following fact follows from the Hamilton's th.

Theorem 2.2.

The solution of (MCFE) for any initial condition uniquely exists in short time.

Hamilton's theorem

N : an $(n+r)$-dimensional Rimannian manifold

$f : M \hookrightarrow N$ an immersion

(W, ϕ) : a local coordinate of N s.t. $f(U) \subset W$
for some open set U of M

An immersion $(\phi \circ f)|_U : U \hookrightarrow \mathbb{R}^{n+r}$ is regarded as a local section of the trivial bundle $M \times \mathbb{R}^{n+r}$ over M.
Then f satisfies (MCFE) if and only if $(\phi \circ f)|_U$ satisfies a parabolic equation for any open subset U of M and any local coordinate (W, ϕ) of N with $f(U) \subset W$.
Hence the statement of Theorem 2.2 follows from the Hamilton's theorem.

3. The evolutions of geometric quantities (hypersurface-case)

The evolutions of geometric quantities (hypersurface-case)

M : an n-dimensional compact manifold

$(N, \langle\ ,\ \rangle)$: an $(n+1)$-dimensional complete Riemannian manifold

$f : M \hookrightarrow N$: an immersion

$f_t\ (0 \leq t < T)$: the mean curvature flow for f

g_t : the induced metric by f_t

ξ_t : a unit normal vector of f_t

h_t : the second fundamental form of f_t (for ξ_t)

A_t : the shape operator of f_t (for ξ_t)

H_t : the mean curvature vector of f_t (for ξ_t)

The evolutions of geometric quantities (hypersurface-case)

ξ : the section of $F^*(TN)$ given by ξ_t's

g : the section of $\pi_M^*(T^{(0,2)}M)$ given by g_t's

h : the section of $\pi_M^*(T^{(0,2)}M)$ given by h_t's

A : the section of $\pi_M^*(T^{(1,1)}M)$ given by A_t's

H : the section of $F^*(TV)$ given by H_t's

The evolutions of geometric quantities (hypersurface-case)

∇^t : the Riemannain connection of g_t

∇ : the connection of $\pi_M^*(TM)$ given by ∇^t's

$$\left(\begin{array}{c} (\nabla_X Y)_{(x,t)} := (\nabla^t_X Y)_x, \quad (\nabla_{\frac{\partial}{\partial t}} Y)_{(x,t)} = \dfrac{dY_{(x,\cdot)}}{dt} \\ \hfill (X,\, Y \in \Gamma(\pi_M^*(TM))) \end{array} \right)$$

Denote by the same symbol ∇ also the connection of $\pi_M^*(T^{(r,s)}M)$ induced from ∇.

$\triangle$: the Laplace op. defined by ∇

$$\left(\begin{array}{c} (\triangle S)_{(x,t)} := \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} S \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))) \\ ((e_1, \cdots, e_n) : \text{ an orthonormal base of } T_x M \text{ w.r.t. } (g_t)_x) \end{array} \right)$$

The evolutions of geometric quantities (hypersurface-case)

R_N : **the curvature tensor of** $(N, \langle\ ,\ \rangle)$

Ric_N : **the Ricci tensor of** $(N, \langle\ ,\ \rangle)$

The evolutions of geometric quantities (hypersurface-case)

Proposition 3.1(Huisken).

- $$\frac{\partial g}{\partial t} = -2||H||h$$
- $$\begin{aligned}\frac{\partial h}{\partial t}(X,Y) = (\triangle h)(X,Y) - 2||H||h(AX,Y)\\ +\mathrm{Tr}\left(A^2 + \mathrm{Ric}_N(\xi,\xi)\right) h(X,Y)\\ -\mathrm{Ric}_N(X,AY) + R_N(X,\xi,\xi,AY)\\ -\mathrm{Ric}_N(Y,AX) + R_N(Y,\xi,\xi,AX)\\ +2\mathrm{Tr}_g^{\bullet} R_N(A\bullet,X,Y,\bullet)\\ -(\nabla^N_Y \mathrm{Ric}_N)(X,\xi) - \mathrm{Tr}(\nabla^N_{\bullet} R_N)(X,\xi,Y)\\ (X,Y\in TM)\end{aligned}$$
- $$\frac{\partial ||H||}{\partial t} = \triangle ||H|| + ||H||\left(\mathrm{Tr}(A^2) + \mathrm{Ric}_N(\xi,\xi)\right)$$

The evolutions of geometric quantities (hypersurface-case)

$$\frac{\partial g}{\partial t} := \nabla_{\frac{\partial}{\partial t}} g$$

$$\mathrm{Tr}_g^{\bullet} S(X, \bullet, Y, \bullet) := \sum_{i=1}^{n} S(X, e_i, Y, e_i)$$

$$((e_1, \cdots, e_n) \ : \ \text{orthon. base of } TM \text{ w.r.t. } g)$$

4. The maximum principle

The maximum principle

$g_t\ (t \in [0,T))$: a C^∞-family of Riemannian metrics of M

g : the section of $\pi_M^*(T^{(0,2)}M)$ given by g_t's

For $B \in \Gamma(\pi_M^*(T^{(r_0,s_0)}M))$, define
$\psi_{B\otimes} : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(r+r_0,s+s_0)}M))$ by

$$\psi_{B\otimes}(S) := B \otimes S \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))).$$

Similarly, define a map $\psi_{\otimes B}$ by

$$\psi_{\otimes B}(S) := S \otimes B \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))).$$

The maximum principle

Define $\psi_{\otimes^k} : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(kr,ks)}M))$ **by**

$$\psi_{\otimes^k}(S) := S\otimes\cdots\otimes S \ (k\text{-times}) \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))).$$

Define $\psi_{g;ij} : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(r,s-2)}M))$ **by**

$$(\psi_{g,ij}(S))_{(x,t)}(X_1,\cdots,X_{s-2}) := \sum_{k=1}^{n} S_{(x,t)}(X_1,\cdots,\underset{i}{e_k},\cdots,\underset{j}{e_k},\cdots,X_{s-2})$$

$$(S \in \Gamma(\pi_M^*(T^{(r,s)}M)),\ \ X_1,\cdots,X_{s-2} \in T_xM),$$

where $\{e_1,\cdots,e_n\}$ **is an orthon. base of** T_xM **w.r.t.** g_t.

The maximum principle

Also, define $\psi_i : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(r-1,s-1)}M))$ **by**

$$\begin{aligned} &(\psi_i(S))_{(x,t)}(X_1,\cdots,X_{s-1}) \\ := &\, \mathrm{Tr}\, S_{(x,t)}(X_1,\cdots,X_{i-1},\bullet,X_i,\cdots,X_{s-1}) \\ &\qquad (S \in \Gamma(\pi_M^*(T^{(r,s)}M)),\ \ X_1,\cdots,X_{s-1} \in T_xM). \end{aligned}$$

The maximum principle

P : a map from $\Gamma(\pi_M^*(T^{(r,s)}M))$ to $\Gamma(\pi_M^*(\bigoplus_{r',s'=0}^{\infty} T^{(r',s')}M))$

Definition(a map of polynomial type).

If P is given as the sum of the compositions of the above five types of maps $\psi_{B\otimes}$, $\psi_{\otimes B}$, $\psi_{\otimes^k}$, $\psi_{g;ij}$, ψ_i, then we say that P is **of polynomial type**.

The maximum principle

Example.

$$P = \psi_{g;5,6} \circ \psi_{g;2,4} \circ \psi_{\otimes^3} + \psi_{g;1,5} \circ \psi_{g;4,6} \circ \psi_3 \circ \psi_{B_1 \otimes} \circ \psi_{\otimes B_2}$$
$$(B_1 \in \Gamma(\pi_M^*(T^{(1,2)}M)),\ B_2 \in \Gamma(\pi_M^*(T^{(0,3)}M)))$$

$$P(S)(X,Y) = \mathrm{Tr}_g^{\bullet_1}\mathrm{Tr}_g^{\bullet_2} S(X,\bullet_1) \otimes S(Y,\bullet_1) \otimes S_{\bullet_2\bullet_2}$$
$$+\mathrm{Tr}\,\mathrm{Tr}_g^{\bullet_1}\mathrm{Tr}_g^{\bullet_2} B_1(\bullet_1, X) \otimes S(\cdot,\bullet_2) \otimes B_2(\bullet_1,\bullet_2,Y)$$

$$P(S)_{i_1 i_2} = g^{k_1 k_2} g^{k_3 k_4} S_{i_1 k_1} S_{i_2 k_2} S_{k_3 k_4}$$
$$+ g^{k_1 k_4} g^{k_3 k_5} (B_1)_{k_1 i_1}^{k_2} S_{k_2 k_3} (B_2)_{k_4 k_5 i_2}$$

The maximum principle

In general, we can define a **multi-variable map P of polynomial type** as a map from $\prod_{i=1}^{k} \Gamma(\pi_M^*(T^{(r_i,s_i)}M))$ to $\Gamma(\pi_M^*(\bigoplus_{r,s=0}^{\infty} T^{(r,s)}M))$.

Remark For $S_i \in \Gamma(\pi_M^*(T^{(r_i,s_i)}M))$ $(i = 1, \cdots, k)$, $P(S_1, \cdots, S_k)$ is described as $S_1 * \cdots * S_k$ in terms of the Hamilton's $*$-notation.

Example.

$$P(S_1, S_2)_{ij} = (S_1)_{k_1k_2}(B_1)_{ij}^{k_1k_2} + (S_2)_{ik_1}(B_2)_j^{k_1} + g^{k_1k_4}g^{k_3k_5}(B_3)_{k_1i}^{k_2}(S_1)_{k_2k_3}(S_2)_{k_4k_5j}$$

The maximum principle

P : a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself

Definition(null vector condition).

Assume that, for any $S \in \Gamma(\pi_M^*(T^{(0,2)}M))$ and any $(x,t) \in M \times [0,T]$,

$$X \in \mathrm{Ker}\, S^\sharp_{(x,t)} \implies P(S)_{(x,t)}(X,X) \geq 0.$$

Then we say that P satisfies the null vector condition.

The maximum principle

P : a map of polynomial type from $\Gamma(\pi_M^*(M \times \mathbb{R}))$ to oneself

Definition(zero point condition).

Assume that, for any $\rho \in \Gamma(\pi_M^*(M \times \mathbb{R}))$ and any $(x, t) \in M \times [0, T)$,

$$\rho(x, t) = 0 \implies P(\rho)(x, t) \geq 0.$$

Then we say that P **satisfies the zero point condition**.

The maximum principle

$g_t\,(0 \leq t < T)$: a C^∞-family of Riemannian metrics on M

$\nabla^t\,(0 \leq t < T)$: the Riemannian connection of g_t

∇ : the connection of $\pi_M^*(TM)$ defined by ∇^t's

The maximum principle

S : an element of $\Gamma(\pi_M^*(T^{(0,2)}M))$

Theorem 4.1(Maximum principle).

Assume that S satisfies

$$\frac{\partial S}{\partial t} = \triangle S + \nabla_{X_0} S + P(S)$$

(• X_0 : an element of $\Gamma(\pi_M^*(TM))$
• P : a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself satisfying the null vector condition)

If $S_0 \geq 0$ (resp. $S_0 > 0$), then $S_t \geq 0$ (resp. $S_t > 0$) holds for all $t \in [0, T)$.

The maximum principle

ρ : an element of $\Gamma(\pi_M^*(M \times \mathbb{R}))$

Theorem 4.2(Maximum principle).

Assume that ρ satisfies

$$\frac{\partial \rho}{\partial t} = \triangle \rho + \nabla_{X_0}\rho + P(\rho)$$

(
- X_0 : an element of $\Gamma(\pi_M^*(TM))$
- P : a map of polynomial type from $\Gamma(\pi_M^*(M \times \mathbb{R}))$ to oneself satisfying the zero point condition

)

If $\rho_0 \geq 0$ (resp. $\rho_0 > 0$), then $\rho_t \geq 0$ (resp. $\rho_t > 0$) holds for all $t \in [0, T)$.

The maximum principle

The maximum principle is used to show the preservability of the geometric properties along the mean curature flow.

5. The mean curvature flow for a convex hypersurface in a Euclidean space

The mean curvature flow for a convex hypersurface in a Euclidean space

$f : M \hookrightarrow \mathbb{R}^{n+1}$: **an embedding** $(\dim M = n)$

$f_t \ (0 \leq t < T)$: **the mean curvature flow for** f

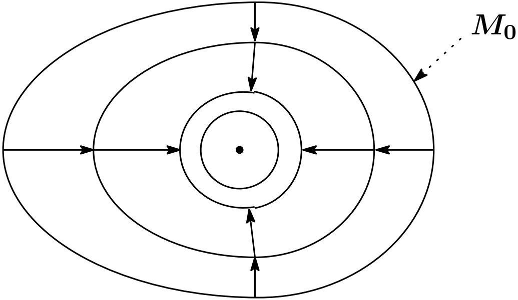
Theorem 5.1(Huisken(JDG-1984)).

Assume that f **is strongly convex (i.e.,** $h > 0$**).**
Then the following statements hold:

(i) $f_t \ (0 \leq t < T)$ **are strongly convex.**

(ii) $\lim_{t \to T} f_t$ **is a constant map.**

The mean curvature flow for a convex hypersurface in a Euclidean space



Assume that f is strongly convex.
Let $(\lim_{t\to T} f_t)(M) = \{p_0\}$.

Definition(The rescaled mean curvature flow)(Huisken).

We define $\widehat{f}_\tau : M \hookrightarrow \mathbb{R}^{n+1} \quad (0 \leq \tau < \infty)$ by

$$\widehat{f}_\tau(x) := \rho(\tau)\left(f_{\phi^{-1}(\tau)}(x) - p_0\right) \quad ((x,\tau) \in M \times [0,\infty)),$$

where ρ is the positive function with $\rho(0) = 1$ such that the volume of $\widehat{f}_\tau(M)$ is constant, ϕ is defined by

$$\tau = \phi(t) := \int_0^t \rho(t)^2 dt.$$

The mean curvature flow for a convex hypersurface in a Euclidean space

Fact.

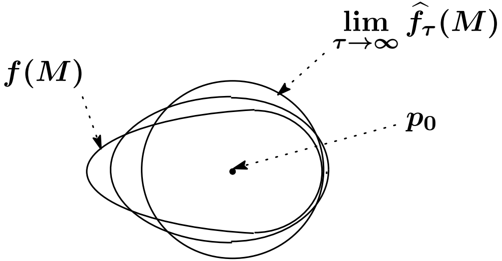
- The rescaled m.c.f. $\widehat{f}_\tau$ $(0 \leq \tau < \infty)$ satisfies

$$\text{(RMCF)} \qquad \frac{\partial \widehat{f}_\tau}{\partial \tau} = \widehat{H}_\tau + \frac{1}{n} H_{\mathrm{sav}} \widehat{f}_\tau,$$

where $\widehat{H}_\tau$ is the mean curvature vector of $\widehat{f}_\tau$ and

$$H_{\mathrm{sav}} := \int_M ||\widehat{H}_\tau||^2 dV_\tau / \mathrm{Vol}(\widehat{f}_\tau(M)).$$

The mean curvature flow for a convex hypersurface in a Euclidean space



$\widehat{f}_\tau$:the rescaled m. c. f.

The mean curvature flow for a convex hypersurface in a Euclidean space

Theorem 5.2(Huisken(JDG-1984)).

If f is strongly convex, then the flow $\widehat{f_\tau}$ converges to a totally umbilic embedding as $\tau \to \infty$.

6. The Outline of the proof of Theorem 5.1

The Outline of the proof of (i) of Theorems 5.1

$f : M \hookrightarrow \mathbb{R}^{n+1}$: strongly convex

$f_t : M \hookrightarrow \mathbb{R}^{n+1}$ $(0 \leq t < T)$: the m.c.f. for f

<u>The outline of the proof of (i) of Theorem 5.1</u>

(Step I) We shall show that $||H_t|| > 0$ holds for all $t \in [0, T)$.

According to Proposition 3.1, we have

$$\frac{\partial ||H||}{\partial t} = \triangle ||H|| + ||H|| \cdot \mathrm{Tr}(A^2).$$

The Outline of the proof of (i) of Theorems 5.1

Define a map P_1 of polynomial type by

$$P_1(\rho) := \rho \cdot \mathrm{Tr}(A^2) \ \ (\rho \in \Gamma(\pi_M^*(M \times \mathbb{R}))).$$

Since P_1 satisfies the zero point condition, it follows from the above evolution eq. and Theorem 4.2 (maximum principle) that

$$||H_t|| > 0 \text{ holds for all } t \in [0, T).$$

The Outline of the proof of (i) of Theorems 5.1

From the assumption, $h_0 - \varepsilon||H_0||g_0 > 0$ holds for some $\varepsilon > 0$.

Set $S_t := h_t - \varepsilon||H_t||g_t$.

(Step II) We shall show that $S_t > 0$ holds for all $t \in [0, T)$.

By using Proposition 3.1, we can show

$$\frac{\partial S}{\partial t} = \triangle S - 2||H||g(A^2(\cdot), \cdot) + ||A||^2 h \\ -\varepsilon H||A||^2 g + 2\varepsilon||H||^2 h.$$

The Outline of the proof of (i) of Theorems 5.1

Let P_2 be the map of polynomial type satisfying

$$P_2(S) = -2||H||g(A^2(\cdot), \cdot) + ||A||^2 h \\ -\varepsilon||H|| \cdot ||A||^2 g + 2\varepsilon||H||^2 h.$$

Since P_2 satisfies the null vector condition, it follows from the above evolution eq. and Theorem 4.1 (maximum principle) that

$$S_t > 0 \text{ holds for all } t \in [0, T).$$

The Outline of the proof of (i) of Theorems 5.1

From $||H_t|| > 0$ and $S_t > 0$ $(t \in [0, T))$, we obtain

$$h_t > 0 \ (t \in [0, T)).$$

This completes the proof of (i) of Theorem 5.1. q.e.d.

The Outline of the proof of (ii) of Theorems 5.1

The outline of the proof of (ii) of Theorem 5.1 by Zhu

(Step I) We shall show $T < \infty$.

S^n : a sphere in $\mathbb{R}^{n+1}$ surrounding $f_0(M)$

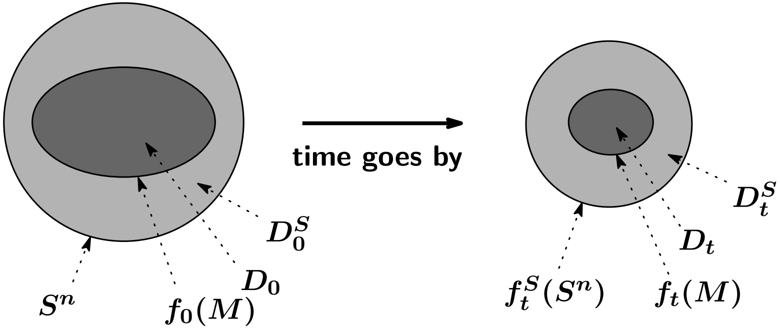
f_t^S : the mean curvature flow for S^n

D_t : the domain surrounded by $f_t(M)$

D_t^S : the domain surrounded by $f_t^S(S^n)$

Then we can show $D_t \subset D_t^S \ (\forall\, t)$.
Also, $f_t^S(S^n)$ collapses to a one-point set in finite time,
From these facts, we obtain $T < \infty$.

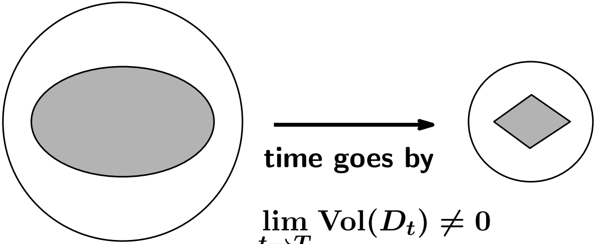
The mean curvature flow for a convex hypersurface in a Euclidean space



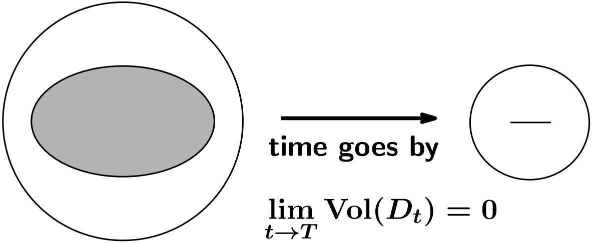
Outline of the proof of (ii) of Theorem 5.1

The following three cases can be considered.

Outline of the proof of (ii) of Theorem 5.1

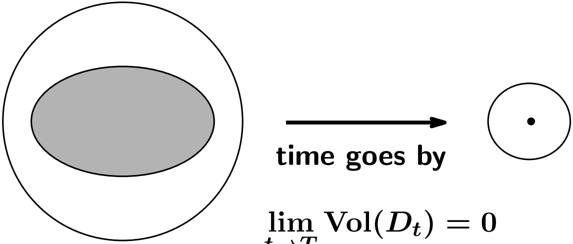


$$\lim_{t \to T} \mathrm{Vol}(D_t) \neq 0$$



$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

Outline of the proof of (ii) of Theorem 5.1



$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

The Outline of the proof of of (ii) Theorems 5.1

(Step II) We shall show that $\lim_{t\to T} \mathrm{Vol}(D_t) = 0$.

Suppose that $\lim_{t\to T} \mathrm{Vol}(D_t) \neq 0$.

Then there exists the ball $B_{r_0}(x_0)$ in $\mathbb{R}^{n+1}$ such that $B_{r_0}(x_0) \subset D_t$ holds for all $t \in [0, T)$.

Without loss of generality, we may assume that x_0 is the origin O of $\mathbb{R}^{n+1}$.

The outline of the proof of (ii) of Theorem 5.1

$S^n(1)$: the unit sphere centered at $x_0 = O$ in $\mathbb{R}^{n+1}$

We define $v : M \times [0, T) \to S^n(1)$ and
$r : S^n(1) \times [0, T) \to \mathbb{R}$ by

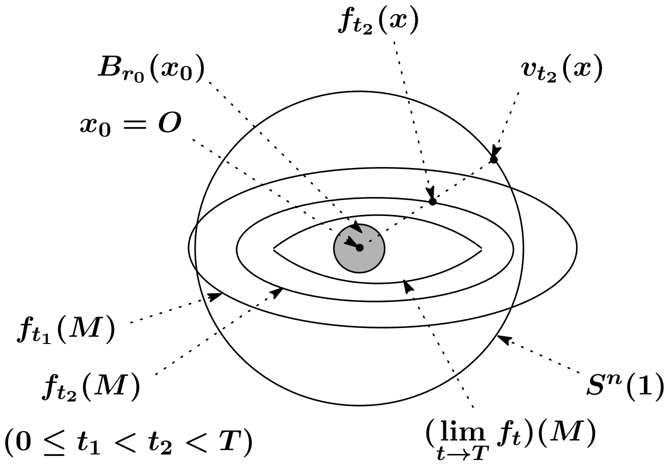
$$f_t(x) = r(v_t(x), t)v_t(x) \quad ((x, t) \in M \times [0, T)).$$

Outline of the proof of (ii) of Theorem 5.1

It is shown that r satisfies the following uniformly parabolic equation with bounded coefficients:

$$\frac{\partial r}{\partial t} = r^{-3}\left(\bar{g}^{ij} - \frac{1}{r^2 + ||\mathrm{grad}\, r||^2}(\mathrm{grad}\, r \otimes \mathrm{grad}\, r)^{ij}\right) \times \left(r(\overline{\nabla} dr)_{ij} - 2(dr \otimes dr)_{ij} - r^2 \bar{g}_{ij}\right)$$

Outline of the proof of (ii) of Theorem 5.1



The outline of the proof of (ii) of Theorem 5.1

Hence, according to the standard regularity theorem for a uniformly parabolic equation (Libermann),

$||\nabla^m r_t||$ is uniformly bounded for any $m \in \mathbb{N}$.

Hence, by the standard discussion based on the Arzela-Ascoli's theorem, we can show that

there exists a seq. $\{t_i\}_{i=1}^{\infty}$ deverging to ∞ s.t. r_{t_i} converges to a smooth function.

The outline of the proof of (ii) of Theorem 5.1

On the other hand, since $B_{r_0}(x_0) \subset f_t(M)$ holds for all $t \in [0, T)$, we obtain $r_{t_i} \geq r_0$.

Therefore it follows that

$$\{\widetilde{f}_{t_i}\}_{i=1}^{\infty} \text{ converges to a smooth embedding.}$$

Furthermore, from the monotonicity of $||f_t(x)||$ w.r.t. t, it follows that

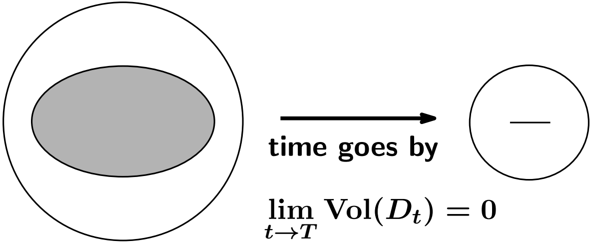
$$f_t \text{ converges to a smooth embedding as } t \to T.$$

This contradicts that T is the explosion time.
Therefore we obtain $\lim_{t \to T} \mathrm{Vol}(D_t) = 0$.

Outline of the proof of (ii) of Theorem 5.1

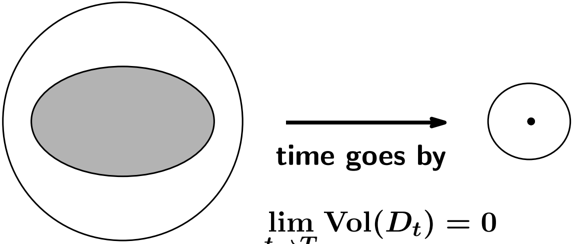
Therefore, the following two cases only can be considered.

Outline of the proof of (ii) of Theorem 5.1



$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

Outline of the proof of (ii) of Theorem 5.1



$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

Outline of the proof of (ii) of Theorem 5.1

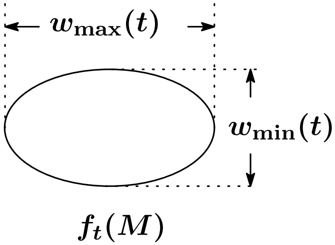
$$\lambda_{\max} : M \times [0,T) \to \mathbb{R}$$
$$\underset{\text{def}}{\Longleftrightarrow} \lambda_{\max}(x,t) := \max \operatorname{Spec} A_x^t \quad ((x,t) \in M \times [0,T))$$

$$\lambda_{\min} : M \times [0,T) \to \mathbb{R}$$
$$\underset{\text{def}}{\Longleftrightarrow} \lambda_{\min}(x,t) := \min \operatorname{Spec} A_x^t \quad ((x,t) \in M \times [0,T))$$

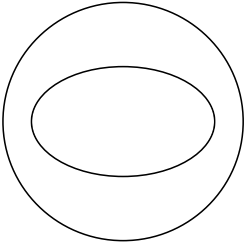
$w_{\max} : [0,T) \to \mathbb{R}$: the maximal width of $f_t(M)$

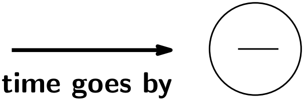
$w_{\min} : [0,T) \to \mathbb{R}$: the minimal width of $f_t(M)$

Outline of the proof of (ii) of Theorem 5.1



Outline of the proof of (ii) of Theorem 5.1





$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

$$\lim_{t \to T} \frac{w_{\max}(t)}{w_{\min}(t)} = \infty$$

The outline of the proof of (ii) of Theorem 5.1

(Step 3) We shall show $\sup_{t\in[0,T)} \frac{w_{\max}(t)}{w_{\min}(t)} < \infty$.

First we show

$$\sup_{(x,t)\in M\times[0,T)} \frac{\lambda_{\max}(x,t)}{\lambda_{\min}(x,t)} < \infty.$$

From this fact, we can show

$$\sup_{t\in[0,T)} \frac{w_{\max}(t)}{w_{\min}(t)} < \infty.$$

Outline of the proof of (ii) of Theorem 5.1

From

$$\lim_{t \to T} \mathrm{Vol}(D_t) = 0$$

and

$$\sup_{t \in [0,T)} \frac{w_{\max}(t)}{w_{\min}(t)} < \infty,$$

it follows that

$f_t(M)$ collapses to the one-point set $\{x_0\}$.

q.e.d.

Uniformly parabolic equation

M : an n-dimensional compact manifold

V : a vector bundle over M

$\Gamma(V)$: the space of all sections of V

E_t : a C^∞-family of diff. op. of order two of V

$\sigma(DE_t)$: the symbol of DE_t

f_t $(0 \leq t < T)$: a C^∞-curve in $\Gamma(V)$

$$F : M \times [0, T) \to V$$

$$\underset{\text{def}}{\Longleftrightarrow} F(x,t) := f_t(x) \ \ ((x,t) \in M \times [0,T))$$

Uniformly parabolic equation

$$(*) \qquad \frac{\partial f_t}{\partial t} = E_t(f_t)$$

If there exists a positive constant C s.t. the real parts of all the eigenvalues of $\sigma(DE_t)(v)$ are greater than $C||v||^2$ for any $v(\neq 0) \in \mathbb{R}^n$ and any $t \in [0, T)$, then the PDE $(*)$ is said to be uniformly parabolic over $M \times [0, T)$.

Outline of the proof of (ii) of Theorem 5.1

The original proof by Huisken

(Step I) We shall show that, for some $\delta \in (0, \frac{1}{2})$,

$$\sup_{0\le t<T} \max_{x\in M} \left(||(A_t)_x||^2 - \frac{1}{n}||(H_t)_x||^2 \right) / ||(H_t)_x||^{2-\delta} < \infty.$$

For $\delta \in (0, \frac{1}{2})$, define a function ρ over $M \times [0, T)$ by

$$\rho(x,t) := \left(||(A_t)_x||^2 - \frac{||(H_t)_x||^2}{n} \right) / ||(H_t)_x||^{2-\delta}.$$

Outline of the proof of Theorem 5.1

Set

$$B_t(k) := \{x \in M \mid \rho_t(x) \geq k\}$$

and

$$||B(k)|| := \int_0^T \int_{B_t(k)} dv_t dt.$$

Outline of the proof of Theorem 5.1

By using the Sobolev inequality for a submanifold (Hoffman-Spruck), the Hölder inequality and the interpolation inequality, we can show that

$$|s_1 - s_2|^{r_1} \cdot ||B(s_1)|| \leq C||B(s_2)||^{r_2}$$

for any s_1, s_2 s.t. $s_1 > s_2 \geq k_0$, where k_0, r_1, r_2 and $C(> 0)$ are some constants. Here we need to choose δ suitably.

Outline of the proof of Theorem 5.1

Hence, by the Stampacchia's iteration lemma, we can show

$$||B(k_0 + d)|| = 0$$

for some constant $d(> 0)$. That is, we obtain

$$\sup_{0 \le t < T} \max_{x \in M} \rho_t(x) \le k_0 + d\,(< \infty).$$

Outline of the proof of Theorem 5.1

(Step II) We show that

$$\lim_{t\to T}\max_{x\in M}||(A_t)_x|| = \infty.$$

Hence we obtain

$$(2) \qquad \lim_{t\to T}\max_{x\in M}||(H_t)_x|| = \infty.$$

Outline of the proof of Theorem 5.1

(Step III) We show that

$$\lim_{t\to T} \frac{\max_{x\in M} \; ||(H_t)_x||}{\min_{x\in M} \; ||(H_t)_x||} = 1. \tag{3}$$

(Step IV) From (1), (2) and (3), we obtain

$$\lim_{t\to T} \left(\frac{||(A_t)_x||}{||(H_t)_x||} - \frac{1}{n} \right) = 0. \tag{4}$$

Outline of the proof of Theorem 5.1

From (2) and (4), it follows that

the diameter of $f_t(M)$ converges to zero as $t \to T$, that is, f_t converges to a constant map.

q.e.d.

7. Sobolev inequality for a submanifold

Sobolev inequality for a submanifold

M : a n-dimensional manifold

N : a $(n+r)$-dimensional Riemnnian manifold

$f : M \hookrightarrow N$: an immersion

H : the mean curvature vector of f

b : the real number or the purely imaginary number s.t. b^2 is equal to the maximum of the sectioal curvatures of N

Assume that $b^2 \neq 0$.

$i(M)$: the injective radius of N restricted to M

ω_n : the volume of the unit ball in $\mathbb{R}^n$

Sobolev inequality for a submanifold

ψ : a non-negative C^1-function over M s.t. $\psi|_{\partial M} = 0$

Fix $\alpha \in (0, 1)$.

ρ_0 : a positive constant described explicitly in terms of b, α and $\mathrm{Vol}(\mathrm{supp}\,\psi)$

Sobolev inequality(Hoffman-Spruck)

If $\rho_0 \leq i(M)$ and $b^2(1-\alpha)^{-2/n}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}\,\psi))^{2/n} \leq 1$, then

$$||\psi||_{L^{n/(n-1)}} \leq C_{n,\alpha} \int_M (||\nabla\,\psi|| + \psi||H||)\,dv.$$

Sobolev inequality for a submanifold

Hölder inequality

Let $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 0, q > 0)$ **and** $\psi_i \in L^p(M) \cap L^q(M)$ $(i = 1, 2)$. **Then**

$$\int_M |\psi_1 \psi_2| dv \leq ||\psi_1||_{L^p} \times ||\psi_2||_{L^q}.$$

Interpolation inequality

Let $1 \leq p < q < r \leq \infty$, $\theta := \frac{1/p - 1/q}{1/p - 1/r}$ **and** $\psi \in L^p(M) \cap L^r(M)$. **Then**

$$||\psi||_{L^q} \leq ||\psi||_{L^p}^{1-\theta} \times ||\psi||_{L^r}^{\theta}.$$

Sobolev inequality for a submanifold

Stambacchia's iteration lemma

Let ψ be a non-negative and non-increasing function over $[a, \infty)$. Assume that

$$\psi(t_2) \leq \left(\frac{C}{t_2 - t_1}\right)^{\alpha} \psi(t_1)^{\beta} \quad (\forall\, t_1, t_2 \text{ s.t. } a < t_1 < t_2),$$

where C, α and β are some constants with $C, \alpha > 0$ and $\beta > 1$. Then we have

$$\psi(a + d) = 0,$$

where $d = C\psi(a)^{(\beta-1)/\alpha} \times 2^{\beta/(\beta-1)}$.

8. The Outline of the proof of Theorem 5.2

The outline of the proof of Theorem 5.2

Outline of the proof of Theorem 5.2 by Zhu

$f : M \hookrightarrow \mathbb{R}^{n+1}$: strongly convex

$\widehat{f}_\tau : M \hookrightarrow \mathbb{R}^{n+1}\ (0 \leq \tau < \infty)$: the rescaled m.c.f. for f

$S^n(1)$: the unit sphere centered at $x_0 = O$ in $\mathbb{R}^{n+1}$

We define $\widehat{v} : M \times [0, \infty) \to S^n(1)$ and
$\widehat{r} : S^n(1) \times [0, \infty) \to \mathbb{R}$ by

$$\widehat{f}_\tau(x) = \widehat{r}(\widehat{v}_\tau(x), \tau)\widehat{v}_\tau(x) \quad ((x, \tau) \in M \times [0, \infty)).$$

The outline of the proof of Theorem 5.2

(Step I) We shall show that

$$\lim_{\tau\to\infty}\widehat{f}_\tau \text{ is a smooth embedding.}$$

First we show that

$\widehat{r}_\tau$ **satisfies a uniformly parabolic equation with bounded coefficients.**

Hence, according to the standard regularity theorem for a uniformly parabolic equation,

$||\nabla^m\widehat{r}_\tau||$ **is uniformly bounded for any** $m\in\mathbb{N}$**.**

The outline of the proof of Theorem 5.2

Hence, by the standard discussion based on the **Arzela-Ascoli's theorem**, we can show that

> there exists a seq. $\{\tau_i\}_{i=1}^{\infty}$ diverging to ∞ s.t. $\widetilde{r}_{\tau_i}$ converges to a smooth function.

Therefore it follows that

> $\{\widehat{f}_{\tau_i}\}_{i=1}^{\infty}$ converges to a smooth embedding.

The outline of the proof of Theorem 5.2

(Step II) We shall show that $\widehat{f}_\infty$ is self-similar.

$$\rho_\tau(x) := \exp(-\tfrac{1}{2}||\widehat{f}_\tau(x)||)$$

First we show the following monotonicity formula:

$$\frac{d}{d\tau}\int_M \rho_\tau d\widehat{v}_\tau \leq -\int_M ||\widehat{H}_\tau + \widehat{f}_\tau \cdot \widehat{\xi}_\tau||\rho_\tau d\widehat{v}_\tau \ (\leq 0),$$

where $\widehat{H}_\tau$, $\widehat{\xi}_\tau$ and $d\widehat{v}_\tau$ are the m. c. v., the u. n. v. f. and the vol. elem. of $\widehat{f}_\tau$, respcetively.

By using this monotonicity formula, it is shown that

$$\lim_{\tau\to\infty}||\widehat{H}_\tau + \widehat{f}_\tau \cdot \widehat{\xi}_\tau|| = 0.$$

This implies that $\widehat{f}_\infty$ is self-similar.

Outline of the proof of Theorem 5.2

According to the classification of the self-similar solution for a compact hypersurface, $\widehat{f}_\infty$ is totally umbilic.

q.e.d.

The outline of the proof of Theorem 5.2

The outline of the proof of Theorem 5.2 by Huisken

$\widehat{h}_\tau$: the second fundamental form of $\widehat{f}_\tau$

$\widehat{A}_\tau$: the shape operator of $\widehat{f}_\tau$

(Step I) First we show that

$$\frac{\max_{x\in M} ||(\widehat{H}_\tau)_x||}{\min_{x\in M} ||(\widehat{H}_\tau)_x||} \to 1 \quad (\tau \to \infty).$$

The outline of the proof of Theorem 5.3

(Step II) By using this fact and discussing deicately, we show that

$$\int_M \left(||\widehat{A}_\tau||^2 - \frac{||\widehat{H_\tau}||^2}{n} \right) d\widehat{v}_\tau \le Ce^{-\delta\tau}.$$

Furthermore, by using the Sobolev inequality and discussing delicately, we show that

$$||\widehat{A}_\tau||^2 - \frac{||\widehat{H_\tau}||^2}{n} \le Ce^{-\delta\tau}.$$

The outline of the proof of Theorem 5.2

Furthermore, by discussing delicately, we show that

$$\max_{x\in M}||(\widehat{H_\tau})_x|| - \min_{x\in M}||(\widehat{H_\tau})_x|| \leq Ce^{-\delta\tau}$$
$$\max_{x\in M}||(\nabla^m \widehat{A}_\tau)_x|| \leq C_m e^{-\delta'\tau} \quad (\forall\, m \in \mathbb{N})$$

From these facts, it follows that

$\widehat{f}_\tau$ converges to a totally umbilic embedding as $\tau \to \infty$.

q.e.d.