The mean curvature flow for a mean convex hypersurface

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8. The mean curvature flow for a convex hypersurface in a Riemannian manifold

The mean curvature flow for a convex hypersurface in a Riemannian manifold

$(N, \langle\ ,\ \rangle)$: an $(n+1)$-dim. complete Riemannian manifold

M : an n-dim. compact manifold

f : an immersion of M into N

$f_t : M \hookrightarrow N\ (0 \le t < T)$: the m.c.f. for f

Assume that

$$-K_1 \le \mathrm{Sec}_N \le K_2 \ \ \&\ \ \|\overline{\nabla}\,\overline{R}\| \le L \ \ \&\ \ i(N) > 0,$$

where Sec_N is the sectional curvature (function) of N, $\overline{\nabla}\,\overline{R}$ is the covariant derivative of the curvature tensor of N, K_1, K_2, L are non-negative constants and $i(N)$ is the injective radius of N.

The mean curvature flow for a convex hypersurface in a Riemannian manifold

Theorem 8.1(Huisken(Invent.M.-1986)).

Assume that

$$||H_0||h_0 > \left(nK_1 + \frac{n^2 L}{||H_0||}\right) g_0.$$

Then the following statements hold:

(i) $||H_t||h_t > \left(nK_1 + \frac{n^2 L}{||H_t||}\right) g_t$ **for all** $t \in [0, T)$.

(ii) f_t **converges to a constant map as** $t \to T$.

The mean curvature flow for a convex hypersurface in a Riemannian manifold

Remark.

(i) **In the case where $(N, \langle\ ,\ \rangle)$ is a symmetric space of compact type, the condition**

$$||H_0||h_0 > \left(nK_1 + \frac{n^2 L}{||H_0||}\right) g_0$$

is equivalent to

$$||H_0||h_0 > 0 \ (\Longleftrightarrow\ h_0 > 0).$$

The mean curvature flow for a convex hypersurface in a Riemannian manifold

(ii) **In the case where $(N, \langle\ ,\ \rangle)$ is a symmetric space of non-compact type, the condition**

$$||H_0||h_0 > \left(nK_1 + \frac{n^2 L}{||H_0||}\right) g_0$$

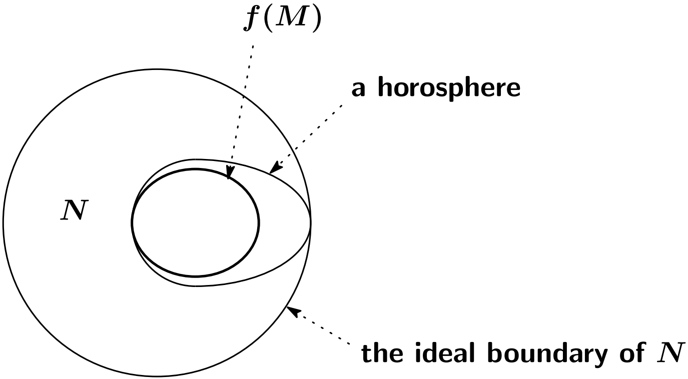
is equivalent to

$$||H_0||h_0 > nK_1 g_0.$$

Here we note that

$$\text{“ } f(M) \ :\ \text{a horosphere} \quad \Longrightarrow \quad ||H_0||h_0 = nK_1 g_0 \text{ ”.}$$

The mean curvature flow for a convex hypersurface in a Riemannian manifold



f satisfies $||H||h > \left(nK_1 + \frac{n^2 L}{||H||}\right) g.$

The mean curvature flow for a convex hypersurface in a Riemannian manifold

According to Theorem 8.1, $f_t(M)$ collapses to a one-point set as $t \to T$.

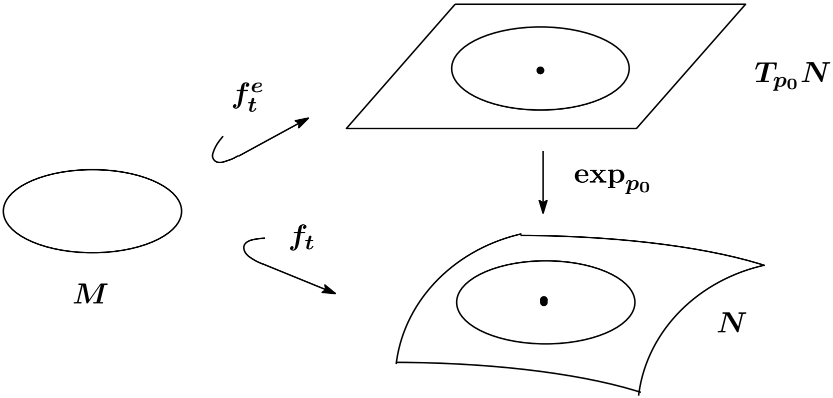
$\{p_0\}$ **: the one-point set.**

$\exp_{p_0}$ **: the exponential map of N at p_0**

f_t^e **: the embedding of M into $T_{p_0}N$ s.t.**

$$\exp_{p_0} \circ f_t^e = f_t$$

The mean curvature flow for a convex hypersurface in a Riemannian manifold



The mean curvature flow for a convex hypersurface in a Riemannian manifold

Definition(The rescaled mean curvature flow).

We define $\widehat{f}_\tau \;:\; M \hookrightarrow N \quad (0 \le \tau < \infty)$ by

$$\widehat{f}_\tau(x) := \exp_{p_0}\left(\rho(\tau) f^e_{\phi^{-1}(\tau)}(x)\right) \qquad ((x,\tau) \in M \times [0,\infty)),$$

where ρ is the positive function with $\rho(0) = 1$ such that the volume of the domain surrounded by $\widehat{f}_\tau(M)$ is constant, and ϕ is defined by

$$\tau = \phi(t) := \int_0^t \rho(t)^2 dt.$$

The mean curvature flow for a convex hypersurface in a Riemannian manifold

Theorem 8.2(Huisken(Invent.M.-1986)).

Assume that

$$\boldsymbol{||H_0||h_0 > \left(nK_1 + \frac{n^2L}{||H_0||}\right)g_0.}$$

Then the rescaled m.c.f. $\widehat{f}_\tau(M)$ **(**$\tau \in [0,\infty)$**) converges to a geodesic sphere centered** p_0 **.**

9. The singularities of two types of the mean curvature flow

The singularities of two types of the mean curvature flow

M : an n-dimensional compact manifold

$(N, \langle \ , \ \rangle)$: an $(n+r)$-dimensional complete Riemannian manifold

f : an immersion of M into N

$f_t : M \hookrightarrow N \ (0 \leq t < T)$: the m.c.f. for f

The singularities of two types of the mean curvature flow

Definition(The singularity of the m.c.f.)

If there exists $x \in M$ such that $\lim_{t \to T} ||(A_t)_x|| = \infty$, then f_t $(0 \leq t < T)$ is said to be of singularity.

In particular, if $\sup_{t \in [0,T)} \left((T-t) \max_{v \in S^{\perp_t} M} ||(A_t)_v||^2 \right) < \infty$, then f_t $(0 \leq t < T)$ is said to be of type I singularity.

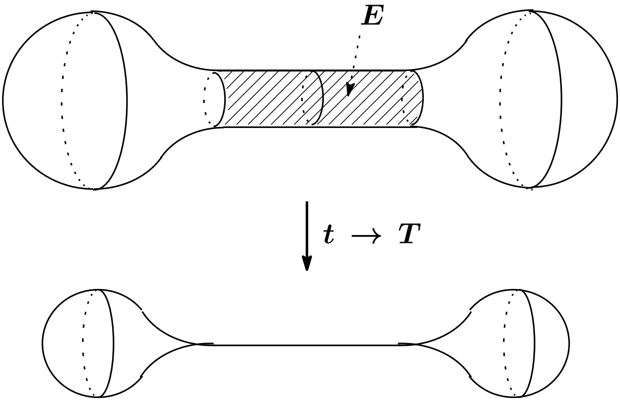
Otherwise, it is said to be of type II singularity.

The singularities of two types of the mean curvature flow

Definition(The blow-up point)

Let $p \in \mathbb{R}^{n+1}$.
If there exists $x \in M$ such that $\lim_{t \to T} ||(A_t)_x|| = \infty$
and that there exists $\lim_{t \to T} f_t(x) = p$,
then p is called the blow-up point of the flow f_t.

The mean curvature flow for a convex hypersurface in a Riemannian manifold



$$\lim_{t\to T} ||(A_t)_x|| = \infty \ \ (x \in E)$$

The singularities of two types of the mean curvature flow

Remark

Assume that f_t $(0 \leq t < T)$ is of singularity. Then, for any $\alpha \in (0,1)$, $\sup_{t \in [0,T)} \left((T-t)^{\alpha} \max_{v \in S^{\perp_t} M} ||(A_t)_v||^2 \right) = \infty$.

10. Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean case)

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean case)

M : an n-dimensional compact manifold

f : an immersion of M into $\mathbb{R}^{n+1}$

$f_t : M \hookrightarrow \mathbb{R}^{n+1}\ (0 \leq t < T)$: the m.c.f. for f

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean case)

Assume that f_t $(0 \leq t < T)$ is of singularity.

Let $p_0 = \lim_{t \to T} f_t(x_0)$ be a blow-up point of the flow f_t.

The mean curvature flow for a convex hypersurface in a Riemannian manifold

Definition(The rescaled mean curvature flow)

Define the flow $\widehat{f}_\tau$ $(0 \leq \tau < \infty)$ by

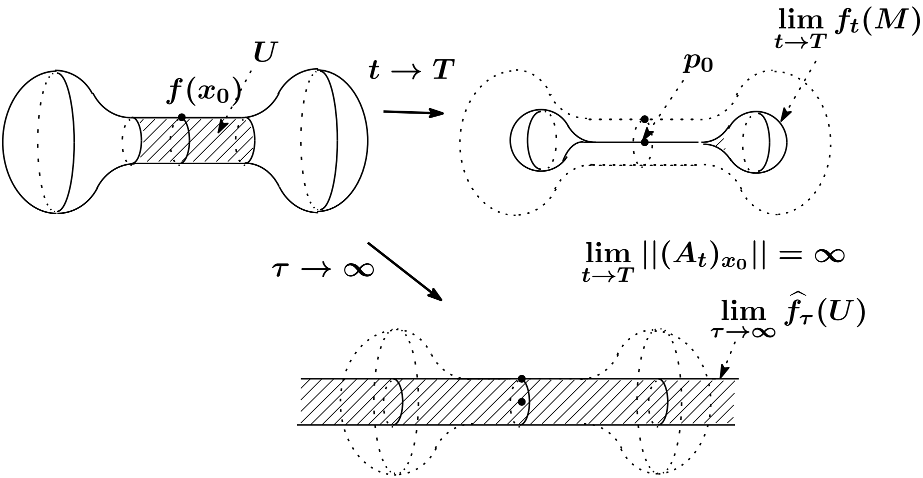
$$\widehat{f}_\tau(x) := \frac{1}{2\sqrt{T - \phi^{-1}(\tau)}}\left(f_{\phi^{-1}(\tau)}(x) - p_0\right) \qquad ((x,\tau) \in M \times [0,\infty)),$$

where ϕ is defined by

$$\phi(t) := -\frac{1}{4}\log\left(\frac{T-t}{T}\right).$$

This flow $\widehat{f}_\tau$ $(0 \leq \tau < \infty)$ is called the rescaled mean curvature flow at the blow-up point p_0.

The mean curvature flow for a convex hypersurface in a Riemannian manifold



Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Theorem 10.1(Huisken(JDG-1990)).

Assume that f is mean convex (i.e., $H > 0$) and that the m.c.f. f_t $(0 \le t < T)$ for f is of type I singularity. Let $p_0 = f_T(x_0)$ be one of blow-up pts of the m.c.f. f_t. Then, for the rescaled m.c.f. $\widehat{f}_\tau$ at p_0, the following (a), (b) or (c) holds:

(a) $\widehat{f}_\infty(M)$ is S^n,

(b) $\widehat{f}_\infty(U)$ is $S^k \times \mathbb{R}^{n-k}$,

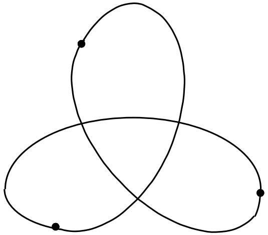
(c) $\widehat{f}_\infty(U)$ is $\gamma \times \mathbb{R}^{n-1}$,

where γ is one of Abresch-Langer curves and U is a neighborhood of p_0 in M.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Abresch-Langer curve means a closed curve γ of positive curvature in $\mathbb{R}^2$ such that the mean curvature flow for γ is self-similar.

Example



Abresch-Langer curve of period 3 and rotation 2

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

The outline of proof

(Step I) We shall show that

> there exists a seq. $\{\tau_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}\tau_i=\infty$ and that $\{\widehat{f}_{\tau_i}\}_{i=1}^{\infty}$ converges to a smooth map.

By a simple calculation, we obtain

$$\frac{\partial||\widehat{\nabla}^m\widehat{A}_\tau||^2}{\partial\tau}\leq\triangle_\tau||\widehat{\nabla}^m\widehat{A}_\tau||^2+C(1+||\widehat{\nabla}^m\widehat{A}_\tau||^2),$$

where C is a constant.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Set

$$\mu_\tau := ||\widehat{\nabla}^m \widehat{A}_\tau||^2 + C||\widehat{\nabla}^{m-1} \widehat{A}_\tau||^2.$$

Then we have

$$\frac{\partial \mu_\tau}{\partial \tau} \leq \widehat{\triangle}_\tau \mu_\tau - C||\widehat{\nabla}^m \widehat{A}_\tau||^2 + C',$$

where C' is a constant.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

By applying the maximum principle to this equation, we have

$$\mu_\tau \leq \mu_0.$$

Hence, if

$$\sup_{(x,\tau)\in M\times[0,\infty)} ||\widehat{\nabla}^{m-1}(\widehat{A}_\tau)_x|| < \infty,$$

then we have

$$\sup_{(x,\tau)\in M\times[0,\infty)} ||\widehat{\nabla}^{m}(\widehat{A}_\tau)_x|| < \infty.$$

Therefore, by the induction, it follows that

$$\sup_{(x,\tau)\in M\times[0,\infty)} ||\widehat{\nabla}^{m}(\widehat{A}_\tau)_x|| < \infty$$

for any $m \in \mathbb{N}$.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Hence, by the standard discussion based on the Arzera-Ascoli's theorem, we can show that

> **there exists a seq. $\{\tau_i\}_{i=1}^{\infty}$ divergenting to ∞ such that $\widehat{f}_{\tau_i}$ converges to a C^{∞}-map $\widehat{f}_{\infty}$.**

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

(Step II) We shall show that

$\widehat{f}_\tau$ converges to a self-similar immersion as $\tau \to \infty$.

Define a function ρ_τ by

$$\rho_\tau(x) := \exp\left(-\frac{1}{2}||\widehat{f}_\tau(x)||^2\right).$$

Then, we can show the following monotonicity formula:

$$\frac{d}{d\tau}\int_M \rho_\tau d\widehat{v}_\tau \leq -\int_M ||\widehat{H}_\tau + \widehat{f}_\tau \cdot \widehat{\xi}_\tau||^2 \rho_\tau d\widehat{v}_\tau \ (\leq 0),$$

where $\widehat{H}_\tau$, $\widehat{\xi}_\tau$ and $d\widehat{v}_\tau$ are the m. c. v., the u. n. v. f. and the vol. elem. of $\widehat{f}_\tau$, respcetively.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

By using this monotonicity formula, it is shown that

$$\lim_{\tau\to\infty}||\widehat{H}_\tau+\widehat{f}_\tau\cdot\widehat{\xi}_\tau||=0$$

and that

$$\{\widehat{f}_\tau\} \text{ converges to the } C^\infty \text{ map } \widehat{f}_\infty.$$

Since $||\widehat{H}_\infty+\widehat{f}_\infty\cdot\widehat{\xi}_\infty||=0$, $\widehat{f}_\infty|_U$ **is a self-similar immersion, where** $U=M$ **or** U **is a neighborhood of** p_0 **in** M.

Type I singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

On the other hand, we can show that $\widehat{f}_\infty|_U$ satisfies

$$(*) \qquad \sup_{x\in U}\frac{||(\widehat{A}_\infty)_x||}{||(\widehat{H}_\infty)_x||}<\infty.$$

Hence, according to to the classification of a self-similar immersions satisfying $(*)$ (Huisken(JDG-1990)), it follows that one of the statements (a), (b) or (c) in Theorem 10.1 holds.

q.e.d.

11. Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean case)

M : an n-dimensional compact manifold

f : an immersion of M into $\mathbb{R}^{n+1}$

$f_t : M \hookrightarrow \mathbb{R}^{n+1}\,(0 \le t < T)$: the m.c.f. for f

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

Take a seq. $\{(x_k, t_k)\}_{k=1}^{\infty}$ in $M \times [0, T)$ s.t. $t_k \leq \frac{1}{k}$ and

$$||(H_{t_k})_{x_k}||^2 \left(T - \frac{1}{k} - t_k\right) = \max_{(x,t)\in M\times[0,T-\frac{1}{k}]} ||(H_t)_x||^2 \left(T - \frac{1}{k} - t\right)$$

Set $C_k := ||(H_{t_k})_{x_k}||^2$.

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

The rescaled m.c.f. $(\widehat{f_k})_\tau : M \to \mathbb{R}^{n+1}$ ($\tau \in (a_k, b_k)$) at $p_k := f_{t_k}(x_k)$ is defined by

$$(\widehat{f_k})_\tau(x) := C_k \left(f_{C_k^{-2}\tau + t_k}(x) - f_{t_k}(x_k) \right) \quad (x \in M).$$

Theorem 11.1(Huisken-Sinestrari(Calc.Var.-1999)).

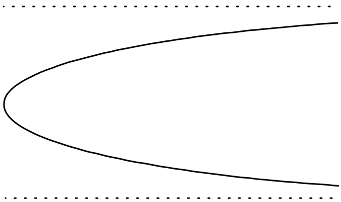
Assume that f_t $(0 \leq t < T)$ is of type II singularity and that f is mean convex.
Then the following statements (i) and (ii) hold.

(i) $\lim\limits_{k\to\infty} a_k = -\infty$ and $\lim\limits_{k\to\infty} b_k = \infty$ holds and, for each $\tau \in (-\infty, \infty)$, the seq. $\{(\widehat{f}_k)_\tau\}_{k=k_\tau}^{\infty}$ has a subseq. $\{(\widehat{f}_{k_i})_\tau\}_{i=1}^{\infty}$ converging to a smooth immersion $(\widehat{f}_\infty)_\tau$, where k_τ is a positive integer s.t. $\tau \in (a_k, b_k)$ $(\forall\, k \geq k_\tau)$. Furthermore, the flow $\widehat{f}_\infty$ $(\tau \in (-\infty, \infty))$ is a mean curvature flow.

(ii) For the limit flow $\widehat{f}_\infty$, the following (a) or (b) hold:
(a) $(\widehat{f}_\infty)_\tau(M)$ is of positive scalar curvature for all τ.
(b) $(\widehat{f}_\infty)_\tau(U)$ is congruent to $\gamma_\tau \times \mathbb{R}^{n-1}$ for all τ, where γ_τ's are grim reapers and U is open in M.

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

A grim reaper is the only translating immersion in $\mathbb{R}^2$.



$$y = \ln \cos x$$

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

The outline of the proof of (i) of Theorem 11.1

First we show the fact similar to the uniformly boundedness of the norms of the higher derivatives of the s.f.f. of $(\widehat{f}_k)_\tau$'s ($k \in \mathbb{N}$), where τ is fixed. Under this fact, we can show that there exists a subseq. $\{(\widehat{f}_{k_i})_\tau\}_{i=1}^\infty$ converging to a smooth immersion $(\widehat{f}_\infty)_\tau$ by the standard discussion based on the Arzela-Ascoli'theorem. Furhtermore, we can show the flow $\widehat{f}_\infty : \tau \mapsto (\widehat{f}_\infty)_\tau$ is a mean curvature flow.

q.e.d.

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

The outline of the proof of (ii) of Theorem 11.1

$\widehat{A}_\tau$: the shape operator of $(\widehat{f}_\infty)_\tau$

$\widehat{H}_\tau$: the mean curvature vector of $(\widehat{f}_\infty)_\tau$

First we show that

$$||\widehat{A}_\tau||^2 \leq ||\widehat{H}_\tau||^2 \ \ (\forall\, \tau).$$

Next we show that

$$||\widehat{A}_\tau||^2 < ||\widehat{H}_\tau||^2 \ \ (\forall\, \tau) \text{ or } ||\widehat{A}_\tau||^2 = ||\widehat{H}_\tau||^2 \ \ (\forall\, \tau).$$

Type II singularity of the mean curvature flow for a mean convex hypersurface (Euclidean-case)

In case of $||\widehat{A}_\tau||^2 < ||\widehat{H}_\tau||^2 \ \ (\forall\,\tau)$**,**

$(\widehat{f}_\infty)_\tau(M)$**'s are of positive scalar curvature.**

In case of $||\widehat{A}_\tau||^2 = ||\widehat{H}_\tau||^2 \ \ (\forall\,\tau)$**,**

$(\widehat{f}_\infty)_\tau(U)$**'s are congruent to** $\Gamma_\tau \times \mathbb{R}^{n-1}$**'s,**

where Γ_τ**'s are grim reaper curves** $y = -\ln \cos x + \tau$**.**

Therefore we obtain (ii) of Theorem 11.1. q.e.d.

12. The evolutions of geometric quantities (higher codimension-case)

The evolutions of geometric quantities (higher codimension-case)

M : an n-dimensional compact manifold

$(N, \langle\ ,\ \rangle)$: an $(n+r)$-dimensional complete Riemannian manifold

$f : M \hookrightarrow N$: an immersion

$f_t\ (0 \leq t < T)$: the mean curvature flow for f

g_t : the induced metric by f_t

h_t : the second fundamental form of f_t (for ξ_t)

A_t : the shape tensor of f_t (for ξ_t)

H_t : the mean curvature vector of f_t (for ξ_t)

The evolutions of geometric quantities (higher codimension-case)

$T^{\perp_t}M$: **the normal bundle of** f_t

$T^{\perp_F}M$: **the subbundle of** F^*TN **defined by**

$$(T^{\perp_F}M)_{(x,t)} := T_x^{\perp_t}M \ \ ((x,t) \in M \times [0,T))$$

The evolutions of geometric quantities (higher codimension-case)

g : **the section of** $\pi_M^*(T^{(0,2)}M)$ **given by** g_t**'s**

h : **the section of** $\pi_M^*(T^{(0,2)}M) \otimes T^{\perp_F}M$ **given by** h_t**'s**

H : **the section of** $T^{\perp_F}M$ **given by** H_t**'s**

A : **the section of** $(T^{\perp_F}M)^* \otimes \pi_M^*(T^{(1,1)}M)$ **given by** A_t**'s**

The evolutions of geometric quantities (higher codimension-case)

$\overline{\nabla}$: the Riemannain connection of $(N, \langle\ ,\ \rangle)$

$\overline{R}$: the curvature tensor of $(N, \langle\ ,\ \rangle)$

∇^t : the Riemannain connection of g_t

∇ : the connection of $\pi_M^*(TM)$ given by ∇^t's

$$\left(\begin{array}{r} (\nabla_X Y)_{(x,t)} := (\nabla^t_X Y)_x, \ \ (\nabla_{\frac{\partial}{\partial t}} Y)_{(x,t)} = \dfrac{dY_{(x,\cdot)}}{dt} \\ (X,\ Y \in \Gamma(\pi_M^*(TM))) \end{array} \right)$$

The evolutions of geometric quantities (higher codimension-case)

$\nabla^{\perp_t}$: the normal connections of f_t

$\nabla^{\perp}$: the connection of $T^{\perp_F}M$ given by $\nabla^{\perp_t}$'s

$$\left(\begin{array}{r} (\nabla^{\perp}_X\xi)_{(x,t)} := (\nabla^{\perp_t}_X\xi_{(\cdot,t)})_x, \ \ (\nabla^{\perp}_{\frac{\partial}{\partial t}}\xi)_{(x,t)} := ((\overline{\nabla}_{\frac{\partial}{\partial t}}\xi)_{\perp})_{(x,t)} \\ (X \in \Gamma(\pi_M^*(TM)), \ \ \xi \in \Gamma(T^{\perp_F}M)) \\ ((\cdot)_{\perp} \ : \ \text{the } T^{\perp_F}M-\text{component of } (\cdot)) \end{array}\right)$$

Remark

$$(F^*TN)_{(x,t)} = (f_t)_*(T_xM) \oplus (T^{\perp_F}M)_{(x,t)}$$

The evolutions of geometric quantities (higher codimension-case)

$\widehat{\nabla}$: **the connection of** $(T^{\perp_F}M)^{(r',s')} \otimes \pi_M^*(T^{(r,s)}M)$ **defined by** ∇ **and** $\nabla^\perp$

$\widehat{\triangle}$: **the Laplace op. defined by** $\widehat{\nabla}$

$$\left(\begin{array}{c} (\widehat{\triangle}S)_{(x,t)} := \sum_{i=1}^{n} \widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i} S \\ (S \in \Gamma((T^{\perp_F}M)^{(r',s')} \otimes \pi_M^*(T^{(r,s)}M))) \\ ((e_1, \cdots, e_n) : \text{ an orthonormal base of } T_xM \text{ w.r.t. } (g_t)_x) \end{array}\right)$$

The evolutions of geometric quantities (higher codimension-case)

Proposition 12.1.

- $$\frac{\partial g}{\partial t}(X,Y) = -2\langle A_H X, Y\rangle$$
- $$\begin{aligned}\frac{\partial h}{\partial t}(X,Y) = {} & (\widehat{\triangle} h)(X,Y) + \mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle h(X,Y),\, h(\bullet,\cdot)\rangle h(\bullet,\cdot) \\ & + \mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle h(Y,\cdot),\, h(\bullet,\cdot)\rangle h(X,\bullet) \\ & - 2\mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle h(X,\bullet),\, h(Y,\cdot)\rangle h(\bullet,\cdot) \\ & + 2\mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle \overline{R}(X,\bullet)Y,\cdot\rangle h(\bullet,\cdot) \\ & - \mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle \overline{R}(\bullet,Y)\bullet,\cdot\rangle h(X,\cdot) \\ & - \mathrm{Tr}_g^{\bullet}\mathrm{Tr}_g^{\cdot}\langle \overline{R}(\bullet,X)\bullet,\cdot\rangle h(Y,\cdot) \\ & + (\mathrm{Tr}_g^{\bullet}(\overline{R}(\bullet,h(X,Y))\bullet))_{\perp} \\ & - 2(\mathrm{Tr}_g^{\bullet}(\overline{R}(X,\bullet)h(Y,\bullet))_{\perp}\end{aligned}$$

The evolutions of geometric quantities (higher codimension-case)

Proposition 12.1(continued).

$$
\begin{aligned}
&-2(\mathrm{Tr}^{\bullet}_{g}(\overline{R}(Y,\bullet)h(X,\bullet))_{\perp}\\
&+(\mathrm{Tr}^{\bullet}_{g}(\overline{\nabla}_{\bullet}\overline{R})(\bullet,X)Y)_{\perp}\\
&-(\mathrm{Tr}^{\bullet}_{g}(\overline{\nabla}_{X}\overline{R})(Y,\bullet)\bullet)_{\perp}
\end{aligned}
$$

$$(X,Y\in TM)$$

- $$
\begin{aligned}
\frac{\partial H}{\partial t} = {} & \widehat{\triangle}\, H + \mathrm{Tr}^{\bullet}_{g}\mathrm{Tr}^{\cdot}_{g}\langle H,\, h(\bullet,\cdot)\rangle h(\bullet,\cdot)\\
&-\mathrm{Tr}^{\bullet}_{g}\mathrm{Tr}^{\cdot}_{g}\mathrm{Tr}^{*}_{g}\langle h(*,\cdot),\, h(\bullet,\cdot)\rangle h(*,\bullet)\\
&+\mathrm{Tr}^{\bullet}_{g}(\overline{R}(\bullet,H)\bullet))_{\perp} - 4\mathrm{Tr}^{\bullet}_{g}\mathrm{Tr}^{\cdot}_{g}(\overline{R}(\cdot,\bullet)h(\cdot,\bullet))_{\perp}
\end{aligned}
$$

13. The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

M : an n-dimensional compact manifold

$f : M \hookrightarrow \mathbb{R}^{n+r}$: an immersion

$f_t \ (0 \leq t < T)$: the mean curvature flow for f

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

Theorem 13.1(Andrews-Baker(JDG-2010)).

Assume that $||H_0|| > 0$ and $||h_0||^2 \leq C_n||H_0||^2$ hold, where C_n is given by

$$C_n := \begin{cases} \dfrac{4}{3n} & (2 \leq n \leq 4) \\ \dfrac{1}{n-1} & (n \geq 4). \end{cases}$$

Then f_t converges to a constant map as $t \to T$.

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

The outline of the proof of Theorem 13.1

(Step I) We show that, for some $\delta \in (0, \frac{1}{2})$,

(1)
$$\sup_{0 \leq t < T} \max_{x \in M} \left(||(A_t)_x||^2 - \frac{1}{n}||(H_t)_x||^2 \right) / ||(H_t)_x||^{2-\delta} < \infty,$$

where we use the assumption

$$||h_0||^2 \leq C_n ||H_0||^2.$$

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

(Step II) We show that

$$\lim_{t\to T}\max_{x\in M}||(A_t)_x||=\infty.$$

Hence we obtain

$$\lim_{t\to T}\max_{x\in M}||(H_t)_x||=\infty. \tag{2}$$

(Step III) We show that

$$\lim_{t\to T}\frac{\max_{x\in M}\ ||(H_t)_x||}{\min_{x\in M}\ ||(H_t)_x||}=1. \tag{3}$$

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

(Step IV) From (1), (2) and (3), we obtain

$$\lim_{t\to T}\left(\frac{||(A_t)_x||}{||(H_t)_x||}-\frac{1}{n}\right)=0. \tag{4}$$

From (2) and (4), it follows that

the diameter of $f_t(M)$ converges to zero as $t \to T$, that is, f_t converges to a constant map.

q.e.d.

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

Assume that $||H_0|| > 0$ **and** $||h_0||^2 \leq C_n||H_0||^2$ **hold.**
Let $(\lim_{t\to T} f_t)(M) = \{p_0\}$.

Definition(The rescaled mean curvature flow).

We define $\widehat{f}_\tau : M \hookrightarrow \mathbb{R}^{n+r} \quad (0 \leq \tau < \infty)$ **by**

$$\widehat{f}_\tau(x) := \frac{1}{\sqrt{2n(T-\phi^{-1}(\tau))}}\left(f_{\phi^{-1}(\tau)}(x) - p_0\right) \quad ((x,\tau) \in M \times [0,\infty)),$$

where ϕ **is defined by**

$$\tau = \phi(t) := -\frac{1}{2n}\log\left(\frac{T-t}{T}\right).$$

The mean curvature flow for a convex hypersurface in a Euclidean space

Theorem 13.2(Andrews-Baker(JDG-2010)).

Assume that $||H_0|| > 0$ and $||h_0||^2 \leq C_n ||H_0||^2$ hold, where C_n is given by

$$C_n := \begin{cases} \dfrac{4}{3n} & (2 \leq n \leq 4) \\ \dfrac{1}{n-1} & (n \geq 4). \end{cases}$$

Then the rescaled mean curvature flow $\widehat{f}_\tau$ converges to a totally umbilic embedding as $\tau \to \infty$.

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

The outline of the proof of Theorem 13.2

$\widehat{h}_\tau$: the second fundamental form of $\widehat{f}_\tau$

$\widehat{A}_\tau$: the shape operator of $\widehat{f}_\tau$

$\widehat{H}_\tau$: the mean curvature vector of $\widehat{f}_\tau$

(Step I) First we show that

$$\frac{\max_{x\in M}\ ||(\widehat{H}_\tau)_x||}{\min_{x\in M}\ ||(\widehat{H}_\tau)_x||} \to 1 \quad (\tau \to \infty).$$

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

(Step II) By using this fact and discussing deicately, we show that

$$\int_M \left(||\widehat{A}_\tau||^2 - \frac{||\widehat{H_\tau}||^2}{n} \right) d\widehat{v}_\tau \leq Ce^{-\delta\tau}.$$

Furthermore, by using the Sobolev inequality and discussing delicately, we show that

$$||\widehat{A}_\tau||^2 - \frac{||\widehat{H_\tau}||^2}{n} \leq Ce^{-\delta\tau}.$$

The mean curvature flow for a submanifold in a Euclidean space (Andrews-Baker's result)

Furthermore, by discussing delicately, we show that

$$\max_{x\in M} ||(\widehat{H}_\tau)_x|| - \min_{x\in M} ||(\widehat{H}_\tau)_x|| \leq Ce^{-\delta\tau}$$
$$\max_{x\in M} ||(\nabla^m \widehat{A}_\tau)_x|| \leq C_m e^{-\delta'\tau} \quad (\forall\, m \in \mathbb{N})$$

From these facts, it follows that

$\widehat{f}_\tau$ converges to a totally umbilic embedding as $\tau \to \infty$.

q.e.d.