# The mean curvature flow for equifocal submanifolds

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# Contents

## 14. The mean curvature flow for an isoparametric submanifold (Liu-Terng's result)

# The mean curvature flow for an isoparametric submanifold

$M$ : an $n$-dimensional manifold

$f : M \hookrightarrow \mathbb{R}^{n+r}$ : an embedding

We identify $M$ with $f(M)$.

**Definition**

$M$ : an isoparametric submanifold

$\underset{\text{def}}{\Longleftrightarrow}$

- the normal holonomy group of $M$ is trivial
- for any parallel normal vec. fd. $v$ of $M$, the principal curvatures for $v_x$ are independent of $x \in M$

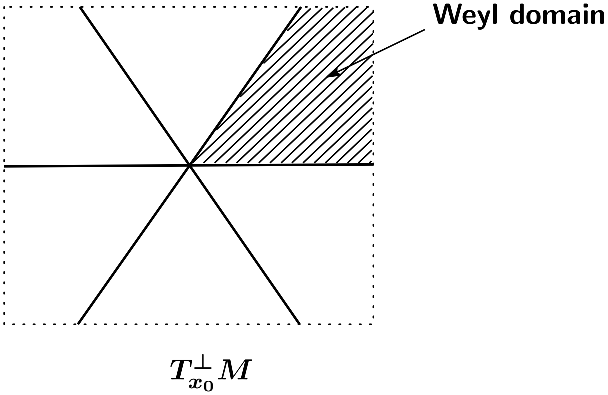
## The mean curvature flow for an isoparametric submanifold

$M$ : an isoparametric submanifold

Fix $x_0 \in M$.

The focal set of $M$ at $x_0$ consists of finite pieces of hyperplanes ($\{l_1, \cdots, l_k\}$) in $T^{\perp}_{x_0}M$.
The reflections w.r.t. $l_i$'s generate a Weyl group.
Fundamental domains of this group are called the Weyl domain of $M$.

# The mean curvature flow for an isoparametric submanifold



# The mean curvature flow for an isoparametric submanifold

$M$ : a compact isoparametric submanifold in $\mathbb{R}^{n+r}$

$M_t\,(0 \leq t < T)$ : the mean curvature flow for $M$

## The mean curvature flow for an isoparametric submanifold

**Theorem 14.1(Liu-Terng(Duke M.J.-2009)).**

**(i)** $M_t\,(0 \le t < T)$ **are parallel submanifolds of** $M$

**(ii)** $T < \infty$

**(iii)** $F := \lim\limits_{t\to T} M_t$ **is a focal submanifold of** $M$

**(iv) If the natural fibration** $\mathrm{pr} : M \to F$ **is spherical,**

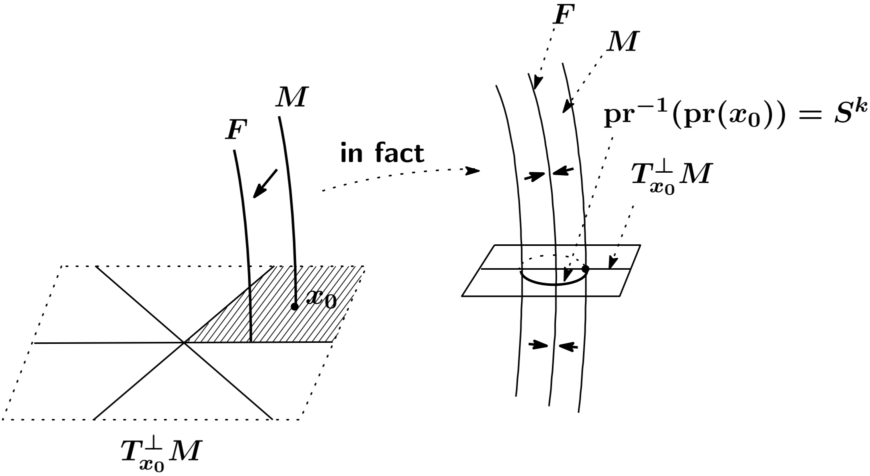
**then** $M_t\,(0 \le t < T)$ **is of type I singularity**

$$\left(\begin{array}{c} \text{i.e.,} \;\sup\limits_{t\in[0,T)} \left((T-t) \max\limits_{v\in S^{\perp}M_t} ||A^t_v||^2\right) < \infty \\ \left(\begin{array}{rl} A^t & : \text{the shape tensor of } M_t \\ S^{\perp}M_t & : \text{the unit normal bd of } M_t \end{array}\right) \end{array}\right)$$

**Remark**

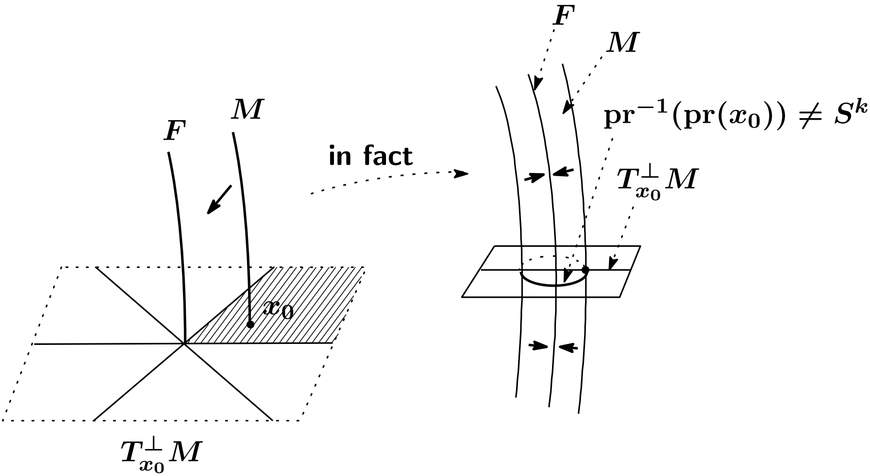
$$\mathrm{pr} : M \to F \underset{\text{def}}{\Longleftrightarrow} \mathrm{pr}(f(x)) := \lim_{t\to T} f_t(x) \;\; (x \in M)$$

# The mean curvature flow for an isoparametric submanifold




$\mathrm{pr} : M \to F$ : spherical

# The mean curvature flow for an isoparametric submanifold




$\mathrm{pr} : M \to F$ : not spherical

# The mean curvature flow for an isoparametric submanifold

**Question 2.**

**How does the m.c. flow for $F$ collapse in the case where $F$ is not minimal ?**

## The mean curvature flow for an isoparametric submanifold

$\widetilde{C}\,(\subset T^{\perp}_{x_0}M)$ : a Weyl domain

$C := \exp^{\perp}(\widetilde{C})(= x_0 + \widetilde{C})$

$\sigma$ : a simplex of $\partial C$ $(\dim \sigma \geq 1)$

$F$ : a focal submanifold of $M$ through $\overset{\circ}{\sigma}$

$F_t\,(0 \leq t < T)$ : the mean curvature flow for $F$

# The mean curvature flow for an isoparametric submanifold

**Theorem 14.2(Liu-Terng(Duke M.J.-2009)).**

**(i)** $F_t\,(0 \leq t < T)$ **are focal submanifolds of** $M$ **thr.** $\overset{\circ}{\sigma}$

**(ii)** $T < \infty$

**(iii)** $F' := \lim_{t \to T} F_t$ **is a focal submanifold of** $M$ **thr.** $\partial\sigma$

**(iv) If the natural fibration** $\mathrm{pr} : F \to F'$ **is spherical**

**then** $F_t\,(0 \leq t < T)$ **is of type I singularity.**

# The mean curvature flow for an isoparametric submanifold

$$\begin{array}{llll} M_t \underset{(t\to T_1)}{\longrightarrow} F^1 & & & \\ & F^1_t \underset{(t\to T_2)}{\longrightarrow} F^2 & & \\ & & \ddots & \\ & & & F^{k-1}_t \underset{(t\to T_k)}{\longrightarrow} \{\mathrm{pt}\} \end{array}$$

$$\left( \begin{array}{l} F^1 \ : \text{ a focal submanifold of } M \\ F^i \ : \text{ a focal submanifold of } F^{i-1} \ (i = 2, \cdots, k-1) \end{array} \right)$$

# 15. The outline of the proof of Liu-Terng's result

# The outline of the proof of Liu-Terng's result

$M$ : **an isoparametric submanifold in** $\mathbb{R}^{n+r}$

$M_t \, (0 \leq t < T)$ : **the mean curvature flow for** $M$

# The outline of the proof of Liu-Terng's result

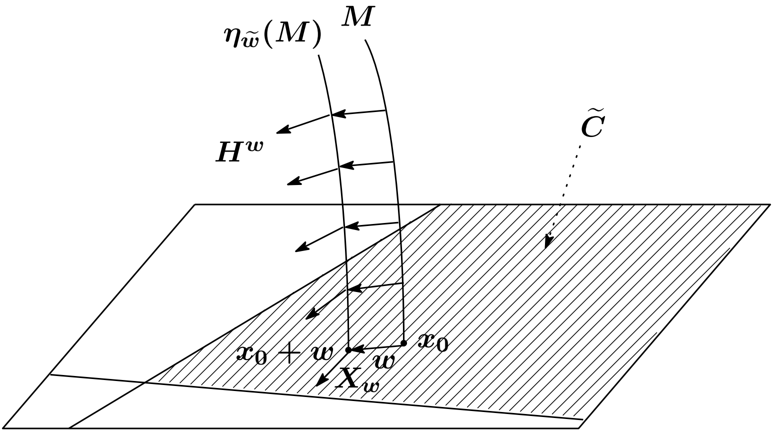
$$x_0 \in M$$

$\widetilde{C}\,(\subset T^{\perp}_{x_0}M)$ : the fund. domain of the Weyl group of $M$ containing $x_0$

**Definition**

$$X \ \text{:a tangent vector field on } \widetilde{C} \underset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} X_w := (H^w)_{x_0+w} \ (w \in \widetilde{C}) \\ \left( \begin{array}{l} H^w \ : \text{the mean curv. vec. of } \eta_{\widetilde{w}}(M) \\ \left( \begin{array}{l} \eta_{\widetilde{w}} \ : \text{the end} - \text{point map for} \\ \qquad \text{a p. n. v. f. } \widetilde{w} \text{ s.t. } \widetilde{w}_{x_0} = w \end{array} \right) \end{array} \right) \end{array} \right.$$

# The outline of the proof of Liu-Terng's result



# The outline of the proof of Liu-Terng's result

**$\{\psi_t\}$ : a local one-parameter transf. gr. of $X$**

**$\xi(t) := \psi_t(0)$ (0 : the zero vector of $T^\perp_{x_0}M$)**

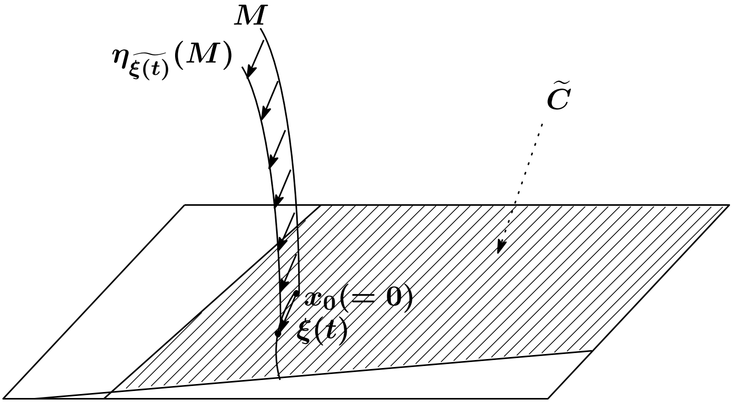
**$\widetilde{\xi(t)}$ : the parallel n.v.f. of $M$ s.t. $\widetilde{\xi(t)}_{x_0} = \xi(t)$**

**Lemma 15.1.**

$$M_t = \eta_{\widetilde{\xi(t)}}(M)$$

**Thus the statement (i) of Theorem 14.1 is shown.**

# The outline of the proof of Liu-Terng's result



# The outline of the proof of Liu-Terng's result

$$X \;---> \;\xi(t) \overset{\textbf{Lem 15.1}}{---->} M_t$$

**Thus we suffice to analyze $X$ in order to analyze the mean curvature flow $M_t$.**

# The outline of the proof of Liu-Terng's result

$A$ : the shape tensor of $M$

$T_xM = \bigoplus_{i\in I_x} E_i^x$ (common eigensp. decomp. of $A_v$'s ($v \in T_x^\perp M$))

$$\lambda_i^x : T_x^\perp M \to \mathbb{R} \underset{\text{def}}{\Longleftrightarrow} A_v|_{E_i^x} = \lambda_i^x(v)\mathrm{id} \quad (v \in T_x^\perp M)$$

**Fact 1.**

$$\lambda_i^x \in (T_x^\perp M)^*$$

# The outline of the proof of Liu-Terng's result

By ordering $E_i^x$'s ($x \in M$) suitably, we may assume that

$$\forall\, i \in I(:= I_x),$$
$$E_i \,:\, x \mapsto E_i^x \ (x \in M) \,:\, C^\infty\text{-distribution}$$

curvature distribution

$$\lambda_i \in \Gamma((T^\perp M)^*) \underset{\text{def}}{\Longleftrightarrow} (\lambda_i)_x := \lambda_i^x \ \ (x \in M)$$

principal curvature

$$\mathbf{n}_i \in \Gamma(T^\perp M) \underset{\text{def}}{\Longleftrightarrow} \lambda_i = \langle \mathbf{n}_i, \cdot \rangle \ \ (i \in I)$$

curvature normal

# The outline of the proof of Liu-Terng's result

**Fact 2.**

$$\bigcup_{i\in I}(\lambda_i)_x^{-1}(1) = \text{"the focal set of } M \text{ at } x\text{"}$$

**Fact 3.**

$$\widetilde{C} = \{w \in T^{\perp}_{x_0}M \mid (\lambda_i)_{x_0}(w) < 1 \quad (i \in I)\}$$

# The outline of the proof of Liu-Terng's result

$$m_i := \dim E_i \ \ (i \in I)$$

**Lemma 15.2.**

$$X_w = \sum_{i \in I} \frac{m_i}{1 - (\lambda_i)_{x_0}(w)} (\mathrm{n}_i)_{x_0}$$

**Remark.**

$$X_w = 0 \iff \eta_{\widetilde{w}}(M) \ : \ \text{minimal}$$

## The outline of the proof of Liu-Terng's result

**Proof of (ii) of Theorem 14.1**

$$\xi_0 \in \bigcap_{i \in I} (\lambda_i)_{x_0}^{-1}(1)$$

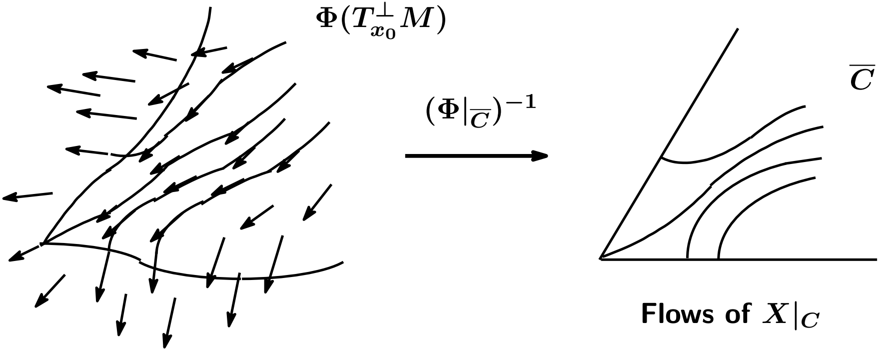
From Lemma 15.2, we have

$$\frac{d}{dt}||\xi(t) - \xi_0||^2 = 2\langle \xi'(t), \xi(t) - \xi_0 \rangle$$
$$= 2\langle X_{\xi(t)}, \xi(t) - \xi_0 \rangle = -2n \quad (n := \dim M)$$

On the other hand, we can show the following fact:

$$\exists\, \Phi \;:\; \text{a polynomial map of } T^{\perp}_{x_0} M \text{ onto } \mathbb{R}^r \;\; (r := \operatorname{codim} M)$$
$$\text{s.t.} \begin{cases} \Phi|_{\overline{\widetilde{C}}}(: \overline{\widetilde{C}} \to \mathbb{R}^r) \;:\; \text{into homeomorphism} \\ \Phi_* X \;:\; \text{a polynomial vec. fd.} \end{cases}$$

# The outline of the proof of Liu-Terng's result




**Flows of $X|_C$**

$Y$

**(The extension of $\Phi_*(X)$)**

## The outline of the proof of Liu-Terng's result

**From these facts, it is shown that**

$\xi(t)$ **converges to a point** $w_1$ **of** $\partial\widetilde{C}$
**as** $t \to T(< \infty)$.

**Since** $M_t = \eta_{\widetilde{\xi(t)}}(M)$ **by Lemma 15.1,**

$M_t$ **collapses to the focal submanifold** $\eta_{\widetilde{w}_1}(M)$
**as** $t \to T(< \infty)$.

**q.e.d.**

# 16. The mean curvature flow for an equifocal submanifold

# The mean curvature flow for an equifocal submanifold

$(N, \langle\ ,\ \rangle)$ : a Riemannian manifold

$M$ : an embedded submanifold in $N$

$\exp^\perp$ : the normal exponential map of $M$

$$v_0 \in T^\perp_{x_0} M$$

**Definition**

$s_0$ : a focal radius along $\gamma_{v_0}$

$\underset{\text{def}}{\Longleftrightarrow}$ $\gamma_{v_0}(s_0)$ : a focal point of $M$ along $\gamma_{v_0}$

$(\Longleftrightarrow$ $(\mathrm{Ker}\ \exp^\perp_{*s_0 v}) \cap (T_{s_0 v}(T^\perp M) \setminus \mathcal{V}_{s_0 v}) \neq \{0\})$

$(\mathcal{V}_{s_0 v}$ : the vertical space of $T^\perp M$ at $s_0 v)$

## The mean curvature flow for an equifocal submanifold

$G/K$ : a symmetric space of compact type

$M$ : an embedded submanifold in $G/K$

**Definition(Terng-Thorbergsson(JDG-1995))**

$M$ : an equifocal submanifold

$\underset{\text{def}}{\Longleftrightarrow}$

- $M$ is compact
- the normal holonomy group of $M$ is trivial
- $M$ has flat section
- for any parallel normal vec. fd. $v$ of $M$, the focal radii along $\gamma_{v_x}$ are indep. of $x \in M$

# The mean curvature flow for an equifocal submanifold

$M$ **has flat section**

$\underset{\text{def}}{\Longleftrightarrow}$ for any $x \in M$, $\Sigma_x := \exp^\perp(T_x^\perp M)$ is totally geodesic and flat.

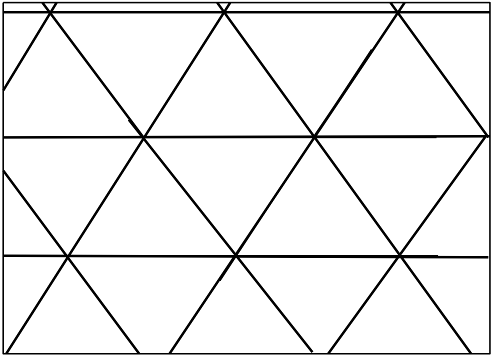
# The mean curvature flow for an equifocal submanifold

$M$ : an equifocal submanifold in $G/K$

$$x_0 \in M$$

The focal set of $M$ at $x_0$ consists of the images of finite pieces of infinite parallel families of hyperplanes ($\mathcal{L}_a := \{l_{ai} \,|\, i \in \mathbb{Z}\}$ $(a = 1, \cdots, k)$) in $T^{\perp}_{x_0}M$ by the normal exponential map.
The reflections w.r.t. $l_{ai}$'s generate a discrete group, that is, a Coxeter group.
This group is called the Coxeter group of $M$.

# The mean curvature flow for an equifocal submanifold



$T^{\perp}_{x_0}M$

## The mean curvature flow for an equifocal submanifold

$G/K$ : a symmetric space of compact type

$M$ : a non-minimal equifocal submanifold in $G/K$

$M_t \, (0 \leq t < T)$ : the mean curvature flow for $M$

## The mean curvature flow for an equifocal submanifold

**Theorem 16.1(K. (Asian J.M.-2011)).**

**(i)** $M_t\,(0 \leq t < T)$ **are parallel submanifolds of** $M$

**(ii)** $T < \infty$

**(iii)** $F := \lim_{t \to T} M_t$ **is a focal submanifold of** $M$

**(iv) If** $M$ **is irreducible, if** $\operatorname{codim} M \geq 2$**,**
**and if the natural fibration** $\mathrm{pr} : M \to F$ **is spherical,**
**then** $M_t\,(0 \leq t < T)$ **is of type I singularity.**

# The mean curvature flow for an equifocal submanifold

**Question.**

**How does the mean curvature flow for $F$ collapse ?**

## The mean curvature flow for an equifocal submanifold

$\widetilde{C}(\subset T^{\perp}_{x_0}M)$ : a fundamental domain (s.t. $0 \in \widetilde{C}$) of the Coxeter group of $M$

$C := \exp^{\perp}(\widetilde{C})$

$\sigma$ : a stratum of $\partial C$ $(\dim \sigma \geq 1)$

$F$ : a non-minimal focal submanifold through $\overset{\circ}{\sigma}$

$F_t \, (0 \leq t < T)$ : the mean curvature flow for $F$

# The mean curvature flow for an equifocal submanifold

**Theorem 16.2(K. (Asian J.M.-2011)).**

**(i)** $F_t\,(0 \leq t < T)$ **are focal submanifolds of** $M$ **through** $\mathring{\sigma}$

**(ii)** $T < \infty$

**(iii)** $F' := \lim_{t\to T} F_t$ **is a focal submanifold of** $M$ **through** $\partial\sigma$

**(iv) If** $M$ **is irreducible, if** $\operatorname{codim} M \geq 2$ **and if the natural fibration** $\mathrm{pr} : F \to F'$ **is spherical, then** $F_t\,(0 \leq t < T)$ **is of type I singularity.**

# The mean curvature flow for an equifocal submanifold

$$M_t \underset{(t\to T_1)}{\longrightarrow} \underset{\text{non}-\text{min.}}{F^1}$$

$$F^1_t \underset{(t\to T_2)}{\longrightarrow} \underset{\text{non}-\text{min.}}{F^2}$$

$$\ddots$$

$$F^{k-1}_t \underset{(t\to T_k)}{\longrightarrow} \underset{\text{min.}}{F^k}$$

$$\left(\begin{array}{l} F^1 \ :\ \text{a focal submanifold of } M \\ F^i \ :\ \text{a focal submanifold of } F^{i-1}\ (i=2,\cdots,k) \end{array}\right)$$

$$\begin{array}{ccc}
\widetilde{M} := (\pi \circ \phi)^{-1}(M) & \hookrightarrow & H^0([0,1], \mathfrak{g}) \\
& & \downarrow \ \phi \\
& & G \\
& & \downarrow \ \pi \\
M & \hookrightarrow & G/K
\end{array}$$

$M$ : **equifocal** $\Leftrightarrow$ $\widetilde{M}$ : **isoparametric**

# 17. Isoparametric submanifolds in a Hilbert space

# Isoparametric submanifolds in a Hilbert space

$V$ : an $\infty$-dimensional (separable) Hilbert space

$f : M \hookrightarrow V$ : an immersion of finite codimension

**Definition(Terng(JDG-1989)).**

$f : M \hookrightarrow V$ : a proper Fredholm submanifold

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ \exp^\perp|_{B^{\perp_1}(M)} \ : \text{proper map} \\ \bullet\ \exp^\perp_{*v} \ : \text{Fredholm op. } (\forall\, v \in T^\perp M) \end{cases}$$

$$\begin{pmatrix} \exp^\perp \ : \text{the normal exponential map of } M \\ B^{\perp_1}(M) \ : \text{the unit normal bnndle map of } M \end{pmatrix}$$

# Isoparametric submanifolds in a Hilbert space

**Fact**

**The shape operator of a proper Fredholm submanifold is compact operator.**

**Definition(Terng(JDG-1989)).**

$f : M \hookrightarrow V$ : **an isoparametric submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet \text{ the normal holonomy group of } M \text{ is trivial} \\ \bullet \text{ For any parallel normal vec. fd. } v \text{ of } M, \\ \quad \text{the principal curvature's for } v_x \\ \quad \text{are independent of } x \in M \end{cases}$$

# Isoparametric submanifolds in a Hilbert space

$f : M \hookrightarrow V$ : an isoparametric submanifold

$$x_0 \in M$$

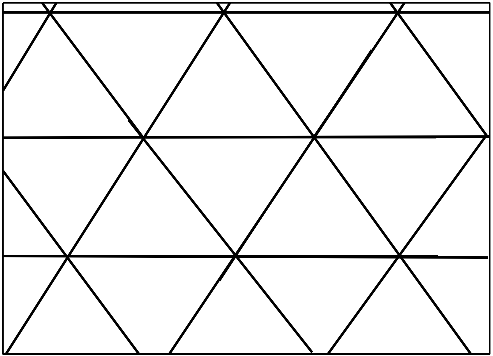
**The focal set of $M$ at $x_0$ consists of finite pieces of infinite parallel families of hyperplanes in $T^{\perp}_{x_0}M$.**

$$(\mathcal{L}_a := \{l_{ai} \,|\, i \in \mathbb{Z}\} \ (a = 1, \cdots, k))$$

**The reflections w.r.t. $l_{ai}$'s generate a discrete group, that is, a Coxeter group.**

**This group is called the Coxeter group of $M$.**

# Isoparametric submanifolds in a Hilbert space



$T^\perp_{x_0}M$

# Isoparametric submanifolds in a Hilbert space

$f : M \hookrightarrow V$ : a proper Fredholm submanifold

**Definition(Heintze-Liu-Olmos(2006)).**

$f : M \hookrightarrow V$ : **a regularizable submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} \forall\, v \in T^{\perp}M, \\ \exists\, \mathrm{Tr}_r A_v \,(<\infty), \quad \exists\, \mathrm{Tr}(A_v^2)\,(<\infty) \\ \left( \begin{array}{l} \mathrm{Tr}_r A_v := \displaystyle\sum_{i=1}^{\infty} (\lambda_i + \mu_i) \\ (\mathrm{Spec}\, A_v = \{\mu_1 \leq \mu_2 \leq \cdot\cdot \leq 0 \leq \cdot\cdot \leq \lambda_2 \leq \lambda_1\}) \\ \mathrm{Tr}(A_v^2) := \displaystyle\sum_{i=1}^{\infty} \nu_i \\ (\mathrm{Spec}\, A_v^2 = \{\nu_1 \geq \nu_2 \geq \cdots > 0\}) \end{array} \right) \end{array} \right.$$

# 18. The mean curvature flows for a regularizable submanifold

# The mean curvature flows for a regularizable submanifold

$V$ : an $\infty$-dimensional (separable) Hilbert space

$f : M \hookrightarrow V$ : a regularizable submanifold

**Definition(Heintze-Liu-Olmos(2006)).**

$$H \underset{\text{def}}{\Longleftrightarrow} \langle H, v \rangle = \mathrm{Tr}_r A_v \ (\forall\, v \in T^{\perp} M)$$

This normal vector field $H$ is called

a regularized mean curvature vector.

# The mean curvature flows for a regularizable submanifold

$f_t : M \hookrightarrow V \ (0 \leq t < T)$ : a $C^\infty$-family of regularizable submanifolds

$$\widetilde{f} : M \times [0,T) \to V$$

$$\underset{\text{def}}{\Longleftrightarrow} \widetilde{f}(x,t) := f_t(x) \ \ ((x,t) \in M \times [0,T))$$

# The mean curvature flows for a regularizable submanifold

**Definition(K. (Asian J.M.-2011))**

$f_t\ (0 \le t < T)$ : **the (regularized) mean curvature flow**

$$\underset{\text{def}}{\Longleftrightarrow} \frac{\partial \widetilde{f}}{\partial t} = H_t \ (0 \le t < T)$$

($H_t$ : the regularized mean curv. vec. of $f_t$)

**Question.**

For any regularlizable submanifold $f$, does the mean curvature flow for $f$ uniquely exist in short time?

## The mean curvature flows for a regularizable submanifold

**In order to solve this question affirmatively, we must show the Hilbert vector bundle version of the Hamilton's theorem for the evolution of a section of a (finite dim.) vector bundle. However, since a regularizable submanifold can be not compact, we must assume a certain kind of compactness for the submanifold.**

# The mean curvature flows for a regularizable submanifold

$G/K$ : a symmetric space of compact type

$M$ : a compact submanifold in $G/K$

$\phi : H^0([0,1],\mathfrak{g}) \to G$ : the parallel transport map for $G$

$$\left( \begin{array}{l} \underset{\text{def}}{\Longleftrightarrow} \ \phi(u) := g_u(1) \ (u \in H^0([0,1],\mathfrak{g})), \\ \qquad \text{where } g_u \text{ is the element of } H^1([0,1],G) \text{ s.t.} \\ \qquad g_u(0) = e \text{ and } (R_{g_u(t)})_*^{-1}(g_u'(t)) = u(t) \ (\forall\, t \in [0,1]) \end{array} \right)$$

$\pi : G \to G/K$ : the natural projection

Set $\widetilde{\phi} := \pi \circ \phi$.

$\widetilde{M} := \widetilde{\phi}^{-1}(M) \ (\hookrightarrow \ H^0([0,1],\mathfrak{g}))$

# The mean curvature flows for a regularizable submanifold

**Fact.**

- $\widetilde{M}$ **is a regularizable submanifold.**
- **There uniquely exists the mean curvature flow** $\widetilde{M}_t$ **for** $\widetilde{M}$ **in short time.**

# 19. The outline of the proof of Theorems 16.1 and 16.2

## The outline of the proof of Theorems 16.1 and 16.2

$M$ : a non-minimal equifocal submanifold in $G/K$

$M_t\,(0 \leq t < T)$ : the mean curvature flow for $M$

## The outline of the proof of Theorems 16.1 and 16.2

**Theorem 16.1(K. (Asian J.M.-2011)).**

**(i)** $M_t\,(0 \leq t < T)$ **are parallel submanifolds of** $M$

**(ii)** $T < \infty$

**(iii)** $F := \lim_{t \to T} M_t$ **is a focal submanifold of** $M$

**(iv) If** $M$ **is irreducible, if** $\operatorname{codim} M \geq 2$**,**
**and if the natural fibration** $\mathrm{pr} : M \to F$ **is spherical,**
**then** $M_t\,(0 \leq t < T)$ **is of type I singularity.**

# The outline of the proof of Theorems 16.1 and 16.2

$$
\begin{array}{ccc}
\widetilde{M} := (\pi \circ \phi)^{-1}(M) & \hookrightarrow & H^0([0,1], \mathfrak{g}) \\
 & & \downarrow \ \phi \\
 & & G \\
 & & \downarrow \ \pi \\
M & \hookrightarrow & G/K
\end{array}
$$

$M$ : equifocal $---->$ $\widetilde{M}$ : regularizable isoparametric

$M_t$ : the mean curvature flow for $M$

$\widetilde{M}_t$ : the mean curvature flow for $\widetilde{M}$

# The outline of the proof of Theorems 16.1 and 16.2

**Lemma 19.1.**

$$\widetilde{M}_t = (\pi \circ \phi)^{-1}(M_t)$$

**According to this fact, the investigation of the flow $M_t$ is reduced to that of the flow $\widetilde{M}_t$.**

## The outline of the proof of Theorems 16.1 and 16.2

$$x_0 \in M$$

$$u_0 \in (\pi \circ \phi)^{-1}(x_0)\ (\subset \widetilde{M})$$

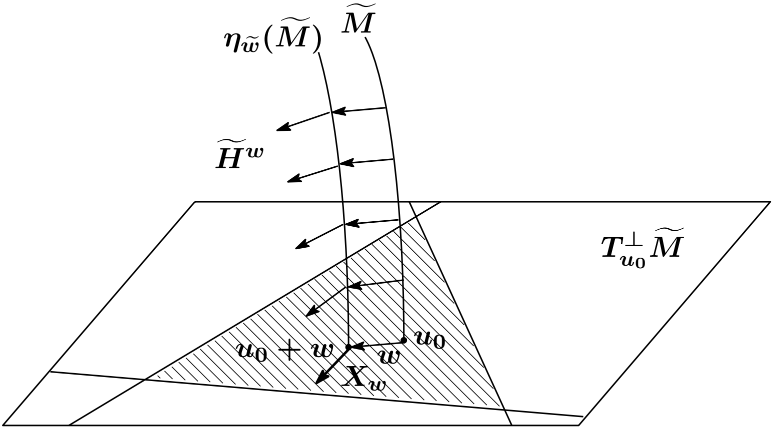
$\widetilde{C}\ (\subset T^{\perp}_{u_0}\widetilde{M})$ : **the fund. domain of the Coxeter group of $\widetilde{M}$ containing $u_0$**

### Definition

$X$ : **a vector field on $\widetilde{C}$**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} X_w := (\widetilde{H}^w)_{u_0+w}\ \ (w \in \widetilde{C}) \\ \left( \begin{array}{l} \widetilde{H}^w\ :\ \text{the reg. mean curv. vec. of } \eta_{\widetilde{w}}(\widetilde{M}) \\ \left( \begin{array}{l} \eta_{\widetilde{w}}\ :\ \text{the end} - \text{point map for} \\ \qquad \text{a p. n. v. f. } \widetilde{w} \text{ s.t. } \widetilde{w}_{u_0} = w \end{array} \right) \end{array} \right) \end{cases}$$

# The outline of the proof of Theorems 16.1 and 16.2



# The outline of the proof of Theorems 16.1 and 16.2

$\{\psi_t\}$ : a local one-parameter transformation gr. of $X$

$\xi(t) := \psi_t(0)$ (0 : the zero vector of $T^{\perp}_{u_0}\widetilde{M}$)

$\widetilde{\xi(t)}$ : the parallel normal vec. fd. of $\widetilde{M}$ s.t. $\widetilde{\xi(t)}_{u_0} = \xi(t)$

**Lemma 19.2.**

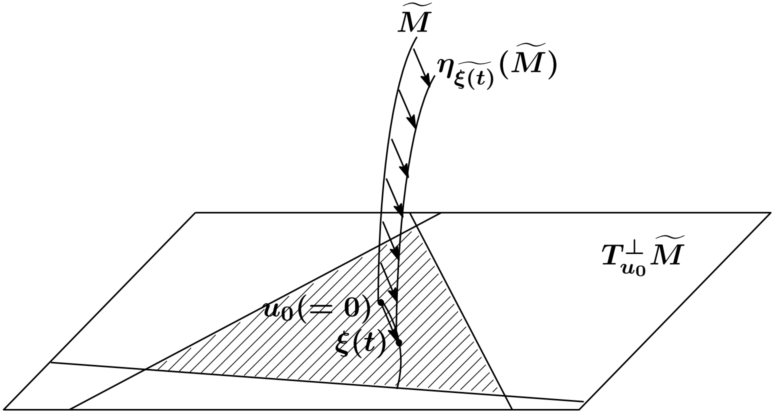
$$\widetilde{M}_t = \eta_{\widetilde{\xi(t)}}(\widetilde{M})$$

**Proof of (i) of Theorem 16.1.**

$$\begin{aligned} M_t = (\pi \circ \phi)(\widetilde{M}_t) &= (\pi \circ \phi)(\eta_{\widetilde{\xi(t)}}(\widetilde{M})) \\ &= \eta_{(\pi \circ \phi)_*(\widetilde{\xi(t)})}(M) \end{aligned}$$

q.e.d.

# The outline of the proof of Theorems 16.1 and 16.2



# The outline of the proof of Theorems 16.1 and 16.2

$$X \; - - - > \; \xi \; \overset{\textbf{Lem 19.2}}{- - - >} \; \widetilde{M}_t \; \overset{\textbf{Lem 19.1}}{- - - >} \; M_t$$

$M$ : equifocal $- - - >$ $\widetilde{M}$ : reg. isoparametric

$\widetilde{A}$ : the shape tensor of $\widetilde{M}$

$T_u\widetilde{M} = \overline{\bigoplus_{i\in I_u} E_i^u}$ (common eigensp. decomp. of $\widetilde{A}_v$'s ($v \in T_u^\perp \widetilde{M}$))

$\lambda_i^u : T_u^\perp \widetilde{M} \to \mathbb{R} \underset{\text{def}}{\Longleftrightarrow} \widetilde{A}_v|_{E_i^u} = \lambda_i^u(v)\mathrm{id} \quad (v \in T_u^\perp \widetilde{M})$

**Fact.**

$$\lambda_i^u \in (T_u^\perp \widetilde{M})^*$$

## The outline of the proof of Theorems 16.1 and 16.2

By choosing $E_i^u$'s $(u \in \widetilde{M})$ suitably, we may assume that

$\forall\, i \in I(:= I_u)$,

$$E_i \,:\, u \mapsto E_i^u \;(u \in \widetilde{M}) \,:\, C^\infty\text{-distribution}$$

curvature distribution

$$\lambda_i \in \Gamma((T^\perp \widetilde{M})^*) \underset{\text{def}}{\Longleftrightarrow} (\lambda_i)_u := \lambda_i^u \;\; (u \in \widetilde{M})$$

principal curvature

$$\mathrm{n}_i \in \Gamma(T^\perp \widetilde{M}) \underset{\text{def}}{\Longleftrightarrow} \lambda_i = \langle \mathrm{n}_i, \cdot \rangle \;\; (i \in I)$$

curvature normal

# The outline of the proof of Theorems 16.1 and 16.2

$\Lambda$ : the set of all principal curvatures of $\widetilde{M}$

## Fact

$$\bigcup_{\lambda\in\Lambda}\lambda_u^{-1}(1) = \text{"the focal set of } \widetilde{M} \text{ at } u\text{"}$$

## Fact

The focal set of $\widetilde{M}$ at $u$ consists of finite pieces of infinite families of parallel hyperplanes in $T_u^{\perp}\widetilde{M}$.

# The outline of the proof of Theorems 16.1 and 16.2

## Fact

The set $\Lambda$ is described as

$$\Lambda = \bigcup_{a=1}^{\bar{r}} \left\{ \frac{\lambda_a}{1 + b_a j} \,\middle|\, j \in \mathbb{Z} \right\}$$

for some $\lambda_a \in \Gamma((T^\perp \widetilde{M})^*)$ and some constant $b_a > 1$ $(a = 1, \cdots, \bar{r})$.

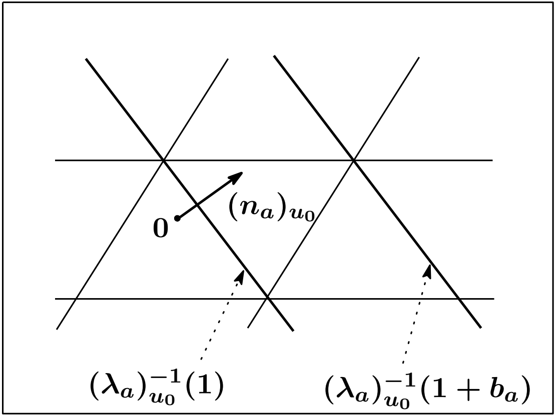
## Fact

$$\widetilde{C} = \{ w \in T^\perp_{u_0} \widetilde{M} \,|\, (\lambda_a)_{u_0}(w) < 1 \quad (a = 1, \cdots, \bar{r}) \}$$

$$E_{aj} := \mathrm{Ker}\left( \widetilde{A}_\cdot - \frac{\lambda_a(\cdot)}{1 + b_a j} \mathrm{id} \right)$$

$$m_a^e := \dim E_{a,2j}, \qquad m_a^o := \dim E_{a,2j+1}$$

# The outline of the proof of Theorems 16.1 and 16.2




$T^{\perp}_{u_0}\widetilde{M}$

# The outline of the proof of Theorems 16.1 and 16.2

## Lemma 19.3.

$$X_w = \sum_{a=1}^{\bar{r}} \left( m_a^e \cot \frac{\pi}{b_a}(1 - (\lambda_a)_{u_0}(w))) - m_a^o \tan \frac{\pi}{b_a}(1 - (\lambda_a)_{u_0}(w)) \right) \frac{\pi}{2b_a}(\mathrm{n}_a)_{u_0}$$

$$\left( \mathrm{n}_a \underset{\mathrm{def}}{\Longleftrightarrow} \langle \mathrm{n}_a, \cdot \rangle = \lambda_a(\cdot) \right)$$

## Remark.

$$X_w = 0 \iff \eta_{\widetilde{w}}(\widetilde{M}) \, : \, \text{minimal}$$

# The outline of the proof of Theorems 16.1 and 16.2

## Proof of (ii) and (iii) of Theorem 16.1

$$\begin{aligned} &\rho \in C^\infty(\widetilde{C}) \\ \underset{\text{def}}{\Longleftrightarrow}\ &\rho(w) := -\sum_{a=1}^{\bar{r}} \left( m_a^e \log\sin\frac{\pi}{b_a}(1-(\lambda_a)_{u_0}(w)) \right. \\ &\left. \qquad + m_a^o \log\cos\frac{\pi}{b_a}(1-(\lambda_a)_{u_0}(w)) \right) \quad (w \in \widetilde{C}) \end{aligned}$$

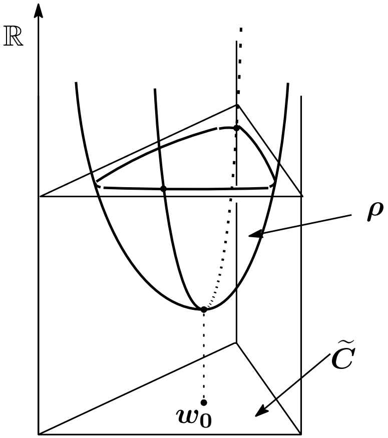
Then we have

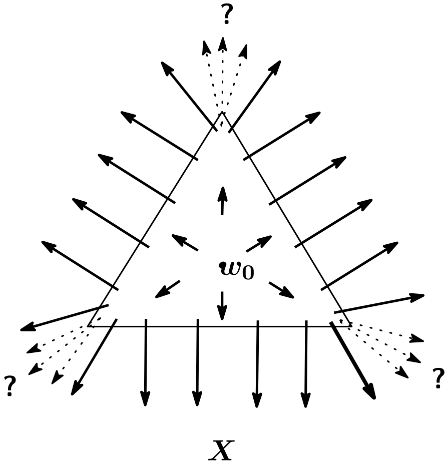
$$\operatorname{grad}\rho = X \quad \text{and} \quad \rho \ : \ \text{downward convex}$$

Also we have

$$\rho(w) \ \to\ \infty \ (w \ \to\ \partial\widetilde{C})$$

# The outline of the proof of Theorems 16.1 and 16.2



## The outline of the proof of Theorems 16.1 and 16.2

**Hence we see that**

**$\rho$ has the only minimal point.**

**Denote by $w_0$ this point. Clearly we have $X_{w_0} = 0$**

**On the other hand, we can show the following fact:**

$$\exists\,\Phi\ :\ \text{a } C^\infty-\text{map of } T^\perp_{u_0}\widetilde{M} \text{ onto } \mathbb{R}^r\ (r := \operatorname{codim} M)$$
$$\text{s.t.}\ \begin{cases} \Phi|_{\overline{\widetilde{C}}}(:\overline{\widetilde{C}} \to \mathbb{R}^r)\ :\ \text{into homeomorphism} \\ \Phi_* X\ :\ \text{a } C^\infty-\text{vec. fd.} \end{cases}$$

# The outline of the proof of Theorems 16.1 and 16.2

From these facts, we see that

- the flow of $X$ starting from a point other than $w_0$ converges to a point of $\partial\widetilde{C}$ in finite time.

Since $\widetilde{M}$ is not minimal, we can show

- $0 \neq w_0$ and the flow $\xi(t)$ of $X$ starting from $0$ converges to a point $w_1$ of $\partial\widetilde{C}$ in finite time $T$.

Since $M_t = \eta_{(\pi\circ\phi)_*(\widetilde{\xi(t)})}(M)$,

$M_t$ collapses to the focal submanifold $\eta_{(\pi\circ\phi)_*(\widetilde{w}_1)}(M)$ in the time $T$.

q.e.d.

## The outline of the proof of Theorems 16.1 and 16.2

$$F := \eta_{(\pi\circ\phi)_*(\widetilde{w}_1)}(M)$$

**(iv) of Theorem 16.1.**

**If $M$ is irreducible, if $\mathrm{codim}\, M \geq 2$,**
**and if the natural fibration $\mathrm{pr} : M \to F$ is spherical,**
**then $M_t\ (0 \leq t < T)$ is of type I singularity.**

$$\left( \text{i.e., } \sup_{t\in[0,T)} \left( (T-t) \max_{v\in S^\perp M_t} ||A^t_v||^2 \right) < \infty \quad \left( \begin{array}{c} A^t \ :\ \text{the shape tensor of } M_t \\ S^\perp M_t \ :\ \text{the unit normal bd of } M_t \end{array} \right) \right)$$

## The outline of the proof of Theorems 16.1 and 16.2

**Proof of (iv) of Theorem 16.1.**

$$\widetilde{F} = \eta_{\widetilde{w}_1}(\widetilde{M})$$

$A^t$ (resp. $\widetilde{A}^t$) : the shape tensor of $M_t$ (resp. $\widetilde{M}_t$)

Since $\mathrm{pr} : M \to F$ is spherical,

$$\exists 1\, a_0 \in \{1, \cdots, \bar{r}\} \text{ s.t. } w_1 \in ((\lambda_{a_0})_{u_0}^{-1}(1) \cap \partial\widetilde{C})^\circ$$

Hence

$$\lim_{t\to T-0} ||\widetilde{A}^t_v||^2_\infty (T-t) = \lim_{t\to T-0} \frac{(\lambda_{a_0})_{u_0}(v)^2}{(1-(\lambda_{a_0})_{u_0}(\xi(t)))^2}(T-t)$$

$$\vdots$$

$$= \frac{(\lambda_{a_0})_{u_0}(v)^2}{2m^e_{a_0}||(\mathrm{n}_{a_0})_{u_0}||^2} \quad \cdots\cdots \quad (1)$$

## The outline of the proof of Theorems 16.1 and 16.2

$M$ : irr. & $\operatorname{codim} M \geq 2$

$--> M$ : curvature-adapted

$$--> \lim_{t\to T-0} ||\widetilde{A}^t_v||^2_\infty (T-t) = \lim_{t\to T-0} ||A^t_{(\pi\circ\phi)_*(v)}||^2_\infty (T-t)$$

$$\cdots\cdots\quad (2)$$

From (1) and (2), we have

$$\lim_{t\to T-0} \max_{v\in S^\perp_{\exp^\perp(\xi(t))} M_t} ||A^t_{(\pi\circ\phi)_*(v)}||^2_\infty (T-t) = \frac{1}{2m^e_{a_0}} < \infty$$

Thus $M_t$ is of type I singularity.

q.e.d.

## The outline of the proof of Theorems 16.1 and 16.2

$\sigma$ : **a stratum of** $\partial C$ **s.t.** $\dim \sigma \geq 1$

$F$ : **a non-minimal focal submanifold of** $M$ **thr.** $\mathring{\sigma}$

$F_t$ : **the mean curvature flow for** $F$

# The outline of the proof of Theorems 16.1 and 16.2

**Theorem 16.2.**

**(i)** $F_t\,(0 \le t < T)$ **are focal submanifolds of** $M$ **through** $\mathring{\sigma}$

**(ii)** $T < \infty$

**(iii)** $F' := \lim_{t\to T} F_t$ **is a focal submanifold of** $M$ **through** $\partial\sigma$

**(iv) If** $M$ **is irreducible, if** $\operatorname{codim} M \ge 2$ **and if the natural fibration** $\mathrm{pr} : F \to F'$ **is spherical, then** $F_t\,(0 \le t < T)$ **is of type I singularity.**

# The outline of the proof of Theorems 16.1 and 16.2

$\widetilde{\sigma}$ : **the simplex of** $\partial\widetilde{C}$ **s.t.** $\exp^{\perp}(\widetilde{\sigma}) = \sigma$

$$w \in (\widetilde{\sigma})^{\circ}$$

$\widetilde{F}_w$ : **the focal submanifold of** $\widetilde{M}$ **through** $w$

**(i.e.,** $\widetilde{F}_w := \eta_{\widetilde{w}}(\widetilde{M})$**)**

$\widetilde{H}^w$ : **the mean curvature vector of** $\widetilde{F}_w$

**Fact.**

$(\widetilde{H}^w)_{u_0+w}$ : **tangent to** $\widetilde{\sigma}$

## Definition

$$X^{\widetilde{\sigma}} \ : \text{a tangent vector field on } \overset{\circ}{\widetilde{\sigma}}$$

$$\underset{\text{def}}{\Longleftrightarrow} \quad X^{\widetilde{\sigma}}_w := (\widetilde{H^w})_{u_0+w} \ (w \in \overset{\circ}{\widetilde{\sigma}})$$

**By analyzing $X^{\widetilde{\sigma}}$, we can show the satements of Theorem 16.2.**