The mean curvature flow for complex equifocal submanifolds

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20. A complex equifocal submanifold

A complex equifocal submanifold

G/K : a symmetric space of non-compact type

$f : M \hookrightarrow G/K$: an embedding

v : a normal vector of f

A_v : the shape operator of f

R : the curvature tensor of G/K $\qquad R(v) := R(\cdot, v)v$

γ_v : the normal geodesic of M with $\gamma_v'(0) = v$

A complex equifocal submanifold

Y_X : (str.) M-Jacobi fd along γ_v s.t. $Y(0) = X\ (\in T_xM)$

Y_X is described as

$$Y_X(s) = P_{\gamma_v|_{[0,s]}}(Q_v(s)X)$$

$$\left(\begin{array}{l} P_{\gamma_v|_{[0,s]}} \ : \ \text{the parallel translation along } \gamma_v|_{[0,s]} \\ Q_v(s) := \cos(s\sqrt{R(v)}) - \dfrac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v \end{array}\right)$$

A complex equifocal submanifold

Definition

$$\begin{array}{rl} & s_0\ (\in \mathbb{R})\ \text{: a focal radius of } M \text{ along } \gamma_v \\ \underset{\text{def}}{\Longleftrightarrow} & \gamma_v(s_0)\ \text{: a focal point of } M \text{ along } \gamma_v \end{array}$$

Fact

Assume that M has flat section. Then

$$\begin{array}{rl} & s_0\ (\in \mathbb{R})\ \text{: a focal radius of } M \text{ along } \gamma_v \\ \Longleftrightarrow & \operatorname{Ker} Q_v(s_0) \neq \{0\} \end{array}$$

A complex equifocal submanifold

Assume that M has flat section.

$$Q_v^{\mathbb{C}}(z) := \cos(z\sqrt{R(v)^{\mathbb{C}}}) - \frac{\sin(z\sqrt{R(v)^{\mathbb{C}}})}{\sqrt{R(v)^{\mathbb{C}}}} \circ A_v^{\mathbb{C}} \quad (z \in \mathbb{C})$$

Definition(K. (Kyushu J.M.-2004))

z_0 $(\in \mathbb{C})$: a **complex focal radius** of M along γ_v

$\underset{\mathrm{def}}{\Longleftrightarrow}$ $\mathrm{Ker}\, Q_v^{\mathbb{C}}(z_0) \neq \{0\}$

A complex equifocal submanifold

Assume that $f : M \hookrightarrow G/K$ **is real analytic.**

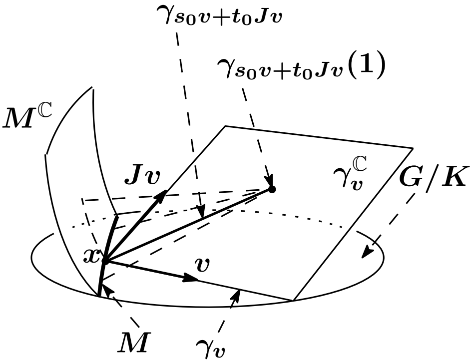
$f^{\mathbb{C}} : M^{\mathbb{C}} \hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}}$ **: the complexification of** f

Fact(K. (Tokyo J.M.-2005))

$z_0\ (\in \mathbb{C})$ **: a complex focal radius of** M **along** γ_v

$\Longleftrightarrow$ $\gamma_v^{\mathbb{C}}(z_0)$ **: a focal point of** $M^{\mathbb{C}}$ **along** $s \mapsto \gamma_v^{\mathbb{C}}(sz_0)$

A complex equifocal submanifold

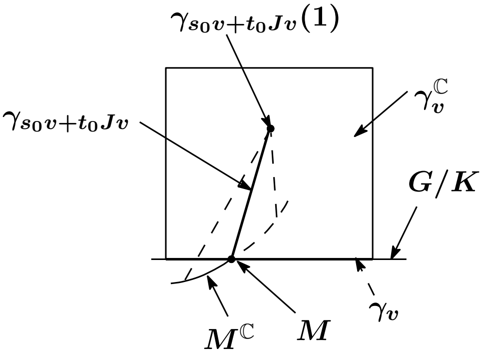


J : the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$

$\gamma_v^{\mathbb{C}} \approx \mathbb{R}^2$ or $\mathbb{R} \times S^1$

$z_0 = s_0 + t_0 \mathrm{i}$: a complex focal radius along γ_v

A complex equifocal submanifold



A complex equifocal submanifold

Definition(K. (Kyushu J.M.-2004))

M : **a complex equifocal submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet \text{ the normal holonomy group of } M \text{ is trivial} \\ \bullet \ M \text{ has flat section} \\ \bullet \text{ for any parallel normal vector field } v, \\ \quad \text{the complex focal radii along } \gamma_{v_x} \text{ is indep. of } x \end{cases}$$

21. A non-Euclidean type focal point on the ideal boundary

A non-Euclidean type focal point on the ideal boundary

v : a unit normal vector of M at x

$\gamma_v(\infty)$: the asymptotic class of the geodesic
$\gamma_v : [0, \infty) \to M$ s.t. $\gamma_v'(0) = v$

$G/K(\infty)$: the ideal boundary of G/K
(i.e., the set of all asymptotic classes of geodesics $\gamma : [0, \infty) \to G/K$'s)

A non-Euclidean type focal point on the ideal boundary

Definition(K. (Kyungpook M.J.-2010))

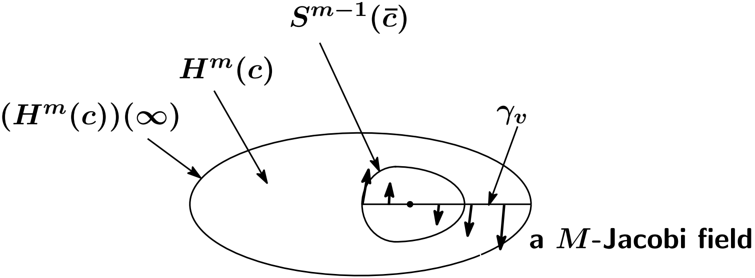
If there exists a M-Jacobi field Y along γ s.t.

$$\lim_{s\to\infty}\frac{||Y(s)||}{s}=0,$$

then we call $\gamma_v(\infty)$ a focal point of M on the ideal boundary $G/K(\infty)$.
In particular, if there exists a M-Jacobi fd Y along γ s.t. $\displaystyle\lim_{s\to\infty}\frac{||Y(s)||}{s}=0$ and $\mathrm{Sec}(v,Y(0))\neq 0$, then we call $\gamma_v(\infty)$ a non-Euclidean type focal point of M on the ideal boundary $G/K(\infty)$.

A non-Euclidean type focal point on the ideal boundary

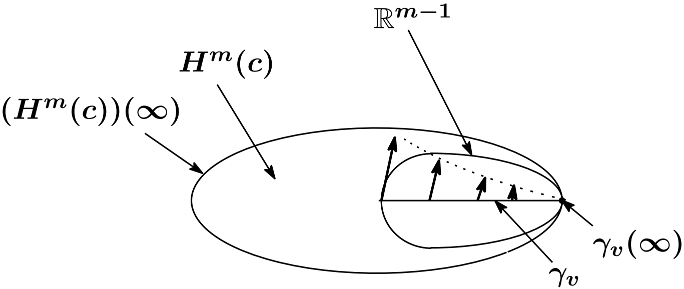
$\underline{M = S^{m-1}(\bar{c}) \subset H^m(c)}$



There exists no focal point on the ideal boundary

A non-Euclidean type focal point on the ideal boundary

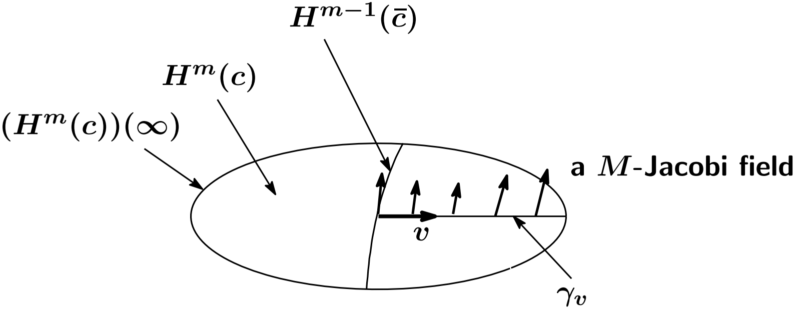
$\underline{M = \mathbb{R}^{m-1} \subset H^m(c)}$



There exists a focal point on the ideal boundary

A non-Euclidean type focal point on the ideal boundary

$\underline{M = H^{m-1}(\bar{c}) \subset H^m(c)}$



There exists no focal point on the ideal boundary

22. The mean curvature flow for a certain kind of complex equifocal submanifold

The mean curvature flow for a certain kind of complex equifocal submanifold

G/K : a symmetric space of non-compact type

M : a curvature-adapted complex equifocal submanifold in G/K admitting no non-Euclidean focal point on the ideal boundary

Assume that M admits focal submanifolds.

F_l : one of the lowest dimensional focal submanifolds of M

Fact.

F_l is a reflective submanifold.

The mean curvature flow for a certain kind of complex equifocal submanifold

Without loss of generelity, we may assume $eK \in F_l$.

$$\mathfrak{p} := T_{eK}(G/K), \quad \mathfrak{p}' := T^{\perp}_{eK}F_l$$

$\mathfrak{b}$: a maximal abelian subspace of $\mathfrak{p}'$

$\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{b}$

$\mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \triangle_+} \mathfrak{p}_\alpha \right)$: the root space decomposition w.r.t. $\mathfrak{a}$

$$\triangle' := \{\alpha|_{\mathfrak{b}} \,|\, \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$$

The mean curvature flow for a certain kind of complex equifocal submanifold

M_t : the mean curvature flow for M

Theorem 22.1(K. (Kodai M.J.-to appear))

Assume that

$\mathrm{codim}\, M = \mathrm{rank}(G/K)$ and $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \geq \frac{1}{2}\dim \mathfrak{p}_\alpha$.

Then the following statements (i) and (ii) hold:

(i) M is not minimal and M_t collapses to a focal submfd F of M in finite time.

(ii) If the natural fibration of M onto F is spherical, then M_t is of type I singularity.

The mean curvature flow for a certain kind of complex equifocal submanifold

Example.

G/K : a symmetric space of non-compact type

θ : involution of G s.t. $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$

H : a symmetric subgroup of G
(i.e., $\exists\,\sigma$: inv. of G s.t. $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$)

$H \curvearrowright G/K$ is called a **Hermann type action**.

The mean curvature flow for a certain kind of complex equifocal submanifold

Fact.

Principal orbits of a Hermann type action are curvature-adapted complex equifocal submanifold admitting no non-Euclidean type focal point on the ideal boundary.

We can find many examlpes of a submanifold satisfying all the conditions in Theorem 22.1 among principal orbits of Hermann type actions.

The mean curvature flow for a certain kind of complex equifocal submanifold

Theorem 22.2(K. (Kodai M.J.-to appear)).

Under the hypothesis of Theorem 22.1, assume that
F_l is a one-point set.
F : a focal submfd of M which is not a one-point set
F_t : the mean curvature flow for F

Then

(i) F is not minimal and F_t collapses to a focal submfd F' of M in finite time.

(ii) If the natural fibration of F onto F' is spherical, then F_t is of type I singularity.

The mean curvature flow for a certain kind of complex equifocal submanifold

$$
\begin{array}{llllll}
M_t \underset{(t\to T_1)}{\longrightarrow} & \underset{\text{non-min.}}{F^1} & & & & \\
& F^1_t \underset{(t\to T_2)}{\longrightarrow} & \underset{\text{non-min.}}{F^2} & & & \\
& & & \ddots & & \\
& & & & F^{k-1}_t \underset{(t\to T_k)}{\longrightarrow} & \{\text{pt.}\}
\end{array}
$$

$$
\left(\begin{array}{l}
F^1\ :\ \text{a focal submanifold of } M \\
F^i\ :\ \text{a focal submanifold of } F^{i-1}\ (i=2,\cdots,k-1)
\end{array}\right)
$$

$$
\begin{array}{ccc}
\widetilde{M} := (\pi \circ \phi)^{-1}(M) & \hookrightarrow & H^0([0,1], \mathfrak{g}) \\
 & & \downarrow \phi \\
 & & G \\
 & & \downarrow \pi \\
M & \hookrightarrow & G/K
\end{array}
$$

M : a curvature-adpated complex equifocal submanifold admitting no non-Euclidean type focal point on the ideal boundary

$\Longrightarrow$ $\widetilde{M}$: reg. proper complex isoparametric submanifold

The mean curvature flow for a certain kind of complex equifocal submanifold

$$
\begin{array}{ccccccccc}
\widetilde{M} & \hookrightarrow & H^0([0,1],\mathfrak{g}) & & & \widetilde{M}^{\mathbb{C}} & \hookrightarrow & H^0([0,1],\mathfrak{g}^{\mathbb{C}}) \\
 & & \downarrow \phi & & & & & \downarrow \phi^{\mathbb{C}} \\
 & & G & & & & & G^{\mathbb{C}} \\
 & & \downarrow \pi & & & & & \downarrow \pi^{\mathbb{C}} \\
M & \hookrightarrow & G/K & \cdots\cdots\cdots\cdots\rightarrow & & M^{\mathbb{C}} & \hookrightarrow & G^{\mathbb{C}}/K^{\mathbb{C}}
\end{array}
$$

23. Proper complex isoparametric submanifolds in a pseudo-Hilbert space

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

V : **an ∞-dimensional pseudo-Hilbert space**

$f : M \hookrightarrow V$: **a Fredholm submanifold**

$$\left(\begin{array}{l}\textbf{i.e., codim}\, M < \infty, \quad f \textbf{ is a proper map} \\ \textbf{and the shape operators are cpt op.} \\ \textbf{for certain kind of inner product}\end{array}\right)$$

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

Definition.

M : **a complex isoparametric submanifold**

$\underset{\text{def}}{\Longleftrightarrow}$

- **the normal holonomy group of M is trivial**
- **for any parallel normal vec. fd. v of M, the complex principal curvatures for v_x are independent of $x \in M$.**

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

Definition

M : **a proper complex isoparametric submanifold**

$\underset{\text{def}}{\Longleftrightarrow}$
- M is complex isoparametric
- for any normal vector v of M, $A_v^{\mathbb{C}}$ is diagonalized with respect to a pseudo-orthonormal base

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

$f : M \hookrightarrow V$: a proper complex isoparametric submfd

$$x_0 \in M$$

The focal set of M at x_0 consists of finite pieces of hyperplanes ($\{l_a \mid a = 1, \cdots, k\}$) in $T^{\perp}_{x_0} M$.
The reflections w.r.t. l_a's generate a discrete group, that is, a Coxeter group. This group is called **the real Coxeter group of M**.

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

Definition

M : **Fredholm submfd with proper shape operators**

$\underset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \bullet\ M \text{ is a Fredholm submanifold} \\ \bullet\ \text{for any normal vector } v \text{ of } M, \\ \quad A_v^{\mathbb{C}} \text{ is diagonalized with respect to} \\ \quad \text{a pseudo-orthonormal base} \end{cases}$

M : **a Fredholm submfd with proper shape operators**

Definition.

M : **regularizable**

$$\underset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} \forall\, v \in T^{\perp}M, \\ \exists\, \mathrm{Tr}_r A_v^{\mathbb{C}}\,(<\infty), \quad \exists\, \mathrm{Tr}(A_v^{\mathbb{C}})^2\,(<\infty) \\ \left(\begin{array}{l} \mathrm{Tr}_r A_v^{\mathbb{C}} := \displaystyle\sum_i m_i \mu_i \\ \left(\begin{array}{l} \mathrm{Spec}\, A_v^{\mathbb{C}} = \{\mu_i \mid i = 1, 2, \cdots\} \\ \left(\begin{array}{l} |\mu_i| > |\mu_{i+1}| \quad \text{or} \\ \text{“}|\mu_i| = |\mu_{i+1}|\ \&\ \mathrm{Re}\,\mu_i > \mathrm{Re}\,\mu_{i+1}\text{”} \\ \text{or “}|\mu_i| = |\mu_{i+1}|\ \&\ \mathrm{Re}\,\mu_i = \mathrm{Re}\,\mu_{i+1} \\ \qquad \&\ \mathrm{Im}\,\mu_i = -\mathrm{Im}\,\mu_{i+1} > 0\text{”} \end{array} \right) \\ m_i \,:\, \text{the multiplicity of } \mu_i \end{array} \right) \end{array} \right) \end{array} \right.$$

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

M : a regularizable submanifold with proper shape perators

Fact

$$\mathbf{Tr}\, A_v^{\mathbb{C}},\ \mathbf{Tr}\,(A_v^{\mathbb{C}})^2 \in \mathbb{R}$$

Definition.

$$H \in \Gamma(T^{\perp}M) \underset{\mathrm{def}}{\Longleftrightarrow} \langle H, v\rangle = \mathrm{Tr}_r A_v^{\mathbb{C}} \ (\forall\, v \in T^{\perp}M)$$

This is called the **regularized mean curvature vector** of M.

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

V : an ∞-dimensional pseudo-Hilbert space

$f : M \hookrightarrow V$: regularizable submanifold with proper shape op.

$f_t : M \hookrightarrow V\ (0 \leq t < T)$: C^∞-family of regularizable submfds with proper shape op.

$$\widetilde{f} : M \times [0,T) \to V$$
$$\underset{\text{def}}{\Longleftrightarrow} \widetilde{f}(x,t) := f_t(x)\ ((x,t) \in M \times [0,T))$$

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

Definition

$f_t \ (0 \le t < T)$: **a mean curvature flow**

$$\underset{\text{def}}{\Longleftrightarrow} \ \frac{\partial \widetilde{f}}{\partial t} = H_t \ (0 \le t < T)$$

(H_t : the regularized mean curv. vec. of f_t)

Question.

For any regularizable submanifold $f : M \hookrightarrow V$ with proper shape op., does the mean curvature flow for f uniquely exist in short time?

Proper complex isoparametric submanifolds in a pseudo-Hilbert space

G/K : a symmtric space of non-compact type

M : a curvature-adapted complex equifocal submanifold in G/K admitting no non-Euclidean type focal point on the ideal boundary

$$\widetilde{M} := (\pi \circ \phi)^{-1}(M) \ (\hookrightarrow \ H^0([0,1], \mathfrak{g}))$$

Fact.

- $\widetilde{M}$ is a reg. proper complex isoparametric submanifold.
- There uniquely exists the mean curvature flow for $\widetilde{M}$ in short time.

24. The outline of the proof of Theorems 22.1 and 22.2

The outline of the proof of Theorems 22.1 and 22.2

M : a curvature-adapted complex equifocal submanifold in G/K admitting no non-Euclidean type focal point on the ideal boundary

Assume that M admits focal submanifolds.

$F_l,\ \triangle,\ \mathfrak{p}_\alpha,\ \mathfrak{p}'$: as in Section 22

M_t : the mean curvature flow for M

The outline of the proof of Theorems 22.1 and 22.2

Theorem 22.1.

Assume that $\mathrm{codim}\, M = \mathrm{rank}(G/K)$ **and that** $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \geq \frac{1}{2}\dim \mathfrak{p}_\alpha$. **Then**

(i) M **is not minimal and** M_t **collapses to a focal submfd** F **of** M **in finite time.**

(ii) If the natural fibration of M **onto** F **is spherical, then** M_t **has type I singularity.**

The outline of the proof of Theorems 22.1 and 22.2

M : **a curvature-adapted proper complex equifocal submanifold as in Theorem 22.1**

M_t : **the mean curvature flow for** M

$\widetilde{M} := (\pi \circ \phi)^{-1}(M)$.

The outline of the proof of Theorems 22.1 and 22.2

Lemma 24.1.

$\widetilde{M}_t := (\pi \circ \phi)^{-1}(M_t)$ **is the mean curvature flow for** $\widetilde{M}$.

The investigation of the flow M_t **is reduced to that of the flow** $\widetilde{M}_t$.

The outline of the proof of Theorems 22.1 and 22.2

$$x_0 \in M$$

$$u_0 \in (\pi \circ \phi)^{-1}(x_0)\ (\subset \widetilde{M})$$

$\widetilde{C}\ (\subset T^{\perp}_{u_0}\widetilde{M})$: the fund. domain of the real Coxeter gr. of $\widetilde{M}$ containing u_0

Definition

X : a tangent vector field on $\widetilde{C}$

$$\underset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} X_w := (\widetilde{H}^w)_{u_0+w} \quad (w \in \widetilde{C}) \\ \left(\begin{array}{l} \widetilde{H}^w \ : \text{the reg. mean curv. vec. of } \eta_{\widetilde{w}}(\widetilde{M}) \\ \left(\begin{array}{l} \eta_{\widetilde{w}} \ : \text{the end} - \text{point map for} \\ \qquad \text{a p. n. v. f. } \widetilde{w} \text{ s.t. } \widetilde{w}_{u_0} = w \end{array} \right) \end{array} \right) \end{array} \right.$$

The outline of the proof of Theorems 22.1 and 22.2

$\{\psi_t\}$: **a local one-parameter transformation gr. of** X

$\xi(t) := \psi_t(0)$ **(**0 **: the zero vector of** $T^{\perp}_{u_0}\widetilde{M}$**)**

$\widetilde{\xi(t)}$ **: the parallel n.v.f. of** $\widetilde{M}$ **s.t.** $\widetilde{\xi(t)}_{u_0} = \xi(t)$

Lemma 24.2.

$$\widetilde{M}_t = \eta_{\widetilde{\xi(t)}}(\widetilde{M})$$

Proof of (i) of Theorem 22.1.

$$\begin{aligned} M_t = (\pi \circ \phi)(\widetilde{M}_t) &= (\pi \circ \phi)(\eta_{\widetilde{\xi(t)}}(\widetilde{M})) \\ &= \eta_{(\pi\circ\phi)_*(\widetilde{\xi(t)})}(M) \end{aligned}$$

q.e.d.

The outline of the proof of Theorems 22.1 and 22.2

$$\boldsymbol{X} \;---> \boldsymbol{\xi} \overset{\textbf{Lem 24.2}}{--->} \widetilde{\boldsymbol{M}_t} \overset{\textbf{Lem 24.1}}{--->} \boldsymbol{M_t}$$

$\boldsymbol{M}$ **: a curv.-adapted complex equifocal submfd admitting no non-Euclidean type focal point on the ideal boundary**

$--> \widetilde{\boldsymbol{M}}$ **: a reg. proper complex isoparametric submfd**

The outline of the proof of Theorems 22.1 and 22.2

$\widetilde{A}$: the shape tensor of $\widetilde{M}$

$(T_u\widetilde{M})^{\mathbb{C}} = \overline{\bigoplus_{i\in I_u} E_i^u}$ (common eigensp. decomp. of $\widetilde{A}_v^{\mathbb{C}}$'s

$(v \in (T_u^{\perp}\widetilde{M})^{\mathbb{C}})$)

$$\lambda_i^u : (T_u^{\perp}\widetilde{M})^{\mathbb{C}} \to \mathbb{C} \underset{\text{def}}{\Longleftrightarrow} \widetilde{A}_v^{\mathbb{C}}|_{E_i^u} = \lambda_i^u(v)\mathrm{id} \quad (v \in T_u^{\perp}\widetilde{M})$$

Fact.

$$\lambda_i^u \in ((T_u^{\perp}\widetilde{M})^{\mathbb{C}})^*$$

The outline of the proof of Theorems 22.1 and 22.2

By ordering E_i^u's $(u \in \widetilde{M})$ suitably, we may assume that

$$\forall\, i \in I(:= I_u),$$
$$E_i \;:\; u \mapsto E_i^u \;\; (u \in \widetilde{M}) \;:\; C^\infty\text{-distribution}$$

complex curvature distribution

$$\lambda_i \in \Gamma(((T^\perp \widetilde{M})^{\mathbb{C}})^*) \underset{\text{def}}{\Longleftrightarrow} (\lambda_i)_u := \lambda_i^u \;\; (u \in \widetilde{M})$$

complex principal curvature

$$\mathbf{n}_i \in \Gamma((T^\perp \widetilde{M})^{\mathbb{C}}) \underset{\text{def}}{\Longleftrightarrow} \lambda_i = \langle \mathbf{n}_i, \cdot \rangle \;\; (i \in I)$$

complex curvature normal

The outline of the proof of Theorems 22.1 and 22.2

Λ : the set of all complex principal curvatures of $\widetilde{M}$

Fact

$$\bigcup_{\lambda\in\Lambda}\lambda_u^{-1}(1) = \text{"the focal set of } \widetilde{M}^{\mathbb{C}} \text{ at } u\text{"}$$

Fact

The focal set of $\widetilde{M}^{\mathbb{C}}$ at u consists of finite pieces of infinite families of parallel complex hyperplanes in $T_u^{\perp}(\widetilde{M}^{\mathbb{C}})$.

The outline of the proof of Theorems 22.1 and 22.2

From these facts, we have

Fact

$$\boldsymbol{\Lambda = \bigcup_{a=1}^{\bar{r}} \left\{ \frac{\lambda_a}{1 + b_a j} \,\middle|\, j \in \mathbb{Z} \right\}}$$
$$\boldsymbol{\left(\lambda_a \in \Gamma(((T^{\perp}\widetilde{M})^{\mathbb{C}})^*), \quad b_a \in \mathbb{C} \text{ s.t. } |b_a| > 1 \right)}$$

The outline of the proof of Theorems 22.1 and 22.2

$$\triangle'^{V}_{+} := \{\beta \in \triangle'_{+} \mid \mathfrak{p}_{\beta} \cap \mathfrak{p}' \neq \{0\}\}$$

$$\triangle'^{H}_{+} := \{\beta \in \triangle'_{+} \mid \mathfrak{p}_{\beta} \cap \mathfrak{p}'^{\perp} \neq \{0\}\}$$

Let $\triangle'_{+} = \{\beta_i \mid i \in I\}, \quad \triangle'^{V}_{+} = \{\beta_i \mid i \in I_{+}\},$

and $\triangle'^{H}_{+} = \{\beta_i \mid i \in I_{-}\}$.

The outline of the proof of Theorems 22.1 and 22.2

From $\mathrm{codim}\, M = \mathrm{rank}(G/K)$ & $\dim(\mathfrak{p}_\alpha \cap \mathfrak{p}') \geq \frac{1}{2}\dim \mathfrak{p}_\alpha$**, we have** $I_- \subset I_+ = I$ **and the following fact:**

Fact

$$\Lambda = \left\{ \frac{\widetilde{\beta_i^{\mathbb{C}}}}{b_i + j\pi\sqrt{-1}} \,\middle|\, i \in I_+ = I,\ j \in \mathbb{Z} \right\} \cup \left\{ \frac{\widetilde{\beta_i^{\mathbb{C}}}}{b_i + (j+\frac{1}{2})\pi\sqrt{-1}} \,\middle|\, i \in I_-,\ j \in \mathbb{Z} \right\}$$

$$\left(\begin{array}{l} \widetilde{\beta_i^{\mathbb{C}}} \ : \text{ the parallel section of } ((T^\perp \widetilde{M})^{\mathbb{C}})^* \\ \qquad\qquad\qquad\qquad\qquad \text{s.t. } (\widetilde{\beta_i^{\mathbb{C}}})_{u_0} = \beta_i^{\mathbb{C}} \\ \qquad\qquad b_i \in \mathbb{R} \end{array} \right)$$

The outline of the proof of Theorems 22.1 and 22.2

Fact

$$\widetilde{C} = \{w \in T^{\perp}_{u_0}\widetilde{M} \mid \beta_i(w) \ < \ b_i \ (i \in I+ = I)\}$$

The outline of the proof of Theorems 22.1 and 22.2

For simplicity, we set

$$\widetilde{\lambda}_{ij}^{+} := \frac{\widetilde{\beta_i^{\mathbb{C}}}}{b_i + j\pi\sqrt{-1}} \quad (i \in I_+ = I,\ j \in \mathbb{Z})$$

$$\widetilde{\lambda}_{ij}^{-} := \frac{\widetilde{\beta_i^{\mathbb{C}}}}{b_i + (j+\frac{1}{2})\pi\sqrt{-1}} \quad (i \in I_-,\ j \in \mathbb{Z})$$

E_{ij}^{+} : the complex curv. distribution corr. to $\widetilde{\lambda}_{ij}^{+}$

E_{ij}^{-} : the complex curv. distribution corr. to $\widetilde{\lambda}_{ij}^{-}$

$m_i^{+} := \dim E_{ij}^{+}, \quad m_i^{-} := \dim E_{ij}^{-}$

The outline of the proof of Theorems 22.1 and 22.2

Lemma 24.3.

$$\boldsymbol{X_w = \sum_{i\in I_+} m_i^+ \coth(b_i - \beta_i(w))\beta_i^\sharp + \sum_{i\in I_-} m_i^- \tanh(b_i - \beta_i(w))\beta_i^\sharp}$$

$$\boldsymbol{\left(\beta_i^\sharp \underset{\text{def}}{\Longleftrightarrow} \langle \beta_i^\sharp, \cdot\rangle = \beta_i(\cdot)\right)}$$

The outline of the proof of Theorems 22.1 and 22.2

Proof of (ii) of Theorem 22.1

$$\begin{aligned} &\rho \in C^\infty(\widetilde{C}) \\ \underset{\text{def}}{\Longleftrightarrow}\ &\rho(w) := -\sum_{i\in I_+} m_i^+ \log\sinh(b_i - \beta_i(w)) \\ &\qquad - \sum_{i\in I_-} m_i^- \log\cosh(b_i - \beta_i(w)) \quad (w \in \widetilde{C}) \end{aligned}$$

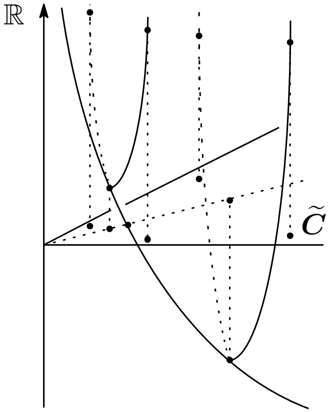
Then we have

$$\operatorname{grad}\rho = X, \qquad \text{and} \qquad \rho : \text{downward convex}$$

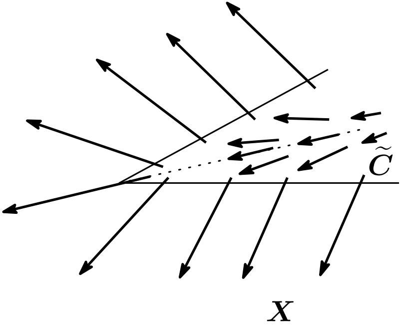
Also we have

$$\rho(w) \to \infty \quad (w \to \partial\widetilde{C})$$

The outline of the proof of Theorems 22.1 and 22.2



the graph of ρ



The outline of the proof of Theorems 22.1 and 22.2

Hence we see that

ρ has no minimal point.

On the other hand, we can show the following fact:

$$\exists\,\Phi\ :\ \text{a polynomial map of } T^{\perp}_{u_0}\widetilde{M} \text{ onto } \mathbb{R}^r\ \ (r := \operatorname{codim} M)$$
$$\text{s.t. } \begin{cases} \Phi|_{\overline{\widetilde{C}}}(:\overline{\widetilde{C}} \to \mathbb{R}^r)\ :\ \text{into homeomorphism} \\ \Phi_* X\ :\ \text{a polynomial vec. fd.} \end{cases}$$

From these facts, we see that

the integral curve $\xi(t)$ of X starting at 0
converges to a pt. w_1 of $\partial\widetilde{C}$ in finite time.

The outline of the proof of Theorems 22.1 and 22.2

Since $\widetilde{M}$ is not minimal,

$0 \neq w_1$ and the flow $\xi(t)$ of X starting 0 converges to a point w_1 of $\partial\widetilde{C}$ in finite time T.

Since $M_t = \eta_{(\pi\circ\phi)_*(\widetilde{\xi(t)})}(M)$,

M_t collapses to the focal submanifold $\eta_{(\pi\circ\phi)_*(\widetilde{w}_1)}(M)$ in the time T. **q.e.d.**

The outline of the proof of Theorems 22.1 and 22.2

Theorem 22.2.

Under the hypothesis of Theorem 22.1, assume that

F_l is a one-point set.

F : a focal submfd of M which is not a one-point set

F_t : the mean curvature flow for F

Then

(i) F is not minimal and F_t collapses to a focal submfd F' of M in finite time.

(ii) If the natural fibration of F onto F' is spherical, then F_t is of type I singularity.

The outline of the proof of Theorems 22.1 and 22.2

σ : the stratum of ∂C passing F

F_t : the mean curv. flow for F

$\widetilde{\sigma}$: the simplex of $\partial\widetilde{C}$ s.t. $\exp^{\perp}(\widetilde{\sigma}) = \sigma$

w_0 : a point of $(\widetilde{\sigma})^{\circ}$

s.t. $\left\{ \begin{array}{l} \exp^{\perp}(w_0) \text{ is the only intersection point} \\ \text{of } F \text{ and } \sigma \end{array} \right.$

The outline of the proof of Theorems 22.1 and 22.2

$\widetilde{F}_w$: **the focal submanifold of** $\widetilde{M}$ **thr.** $w \in (\widetilde{\sigma})^\circ$
(i.e., $\widetilde{F}_w := \eta_{\widetilde{w}}(\widetilde{M})$**)**

$\widetilde{H}^w$: **the reg. mean curvature vector of** $\widetilde{F}_w$

Fact.

$(\widetilde{H}^w)_{u_0+w}$: **tangent to** $(\widetilde{\sigma})^\circ$

Definition

$X^{\widetilde{\sigma}}$: **a tang. vec. fd. on** $(\widetilde{\sigma})^\circ$

$\underset{\text{def}}{\Longleftrightarrow}$ $X^{\widetilde{\sigma}}_w := (\widetilde{H}^w)_{u_0+w} \ (w \in (\widetilde{\sigma})^\circ)$

The outline of the proof of Theorems 22.1 and 22.2

$$I_+^{w_0} := \{i \in I_+(= I) \mid \beta_i(w_0) = b_i\}$$

Since F in not the lowest-dim. focal submfd of M, we have

$$I_+ \setminus I_+^{w_0} \neq \emptyset$$

Since F_l is a one-point set, we have

$$I_- = \emptyset$$

Lemma 24.4.

$$X_w^{\widetilde{\sigma}} = \sum_{i \in I_+ \setminus I_+^{w_0}} m_i^+ \coth(b_i - \beta_i(w))\beta_i^\sharp \quad (w \in (\widetilde{\sigma})^\circ)$$

$$\left(\beta_i^\sharp \underset{\text{def}}{\Longleftrightarrow} \langle \beta_i^\sharp, \cdot \rangle = \beta_i(\cdot)\right)$$

The outline of the proof of Theorems 22.1 and 22.2

Proof of (i) of Theorem 22.2

$$\rho^{\widetilde{\sigma}} \in C^{\infty}(\widetilde{\sigma})$$

$$\underset{\text{def}}{\Longleftrightarrow} \rho^{\widetilde{\sigma}}(w) := -\sum_{i \in I_+ \setminus I_+^{w_0}} m_i^+ \log \sinh(b_i - \beta_i(w))$$

$$(w \in (\widetilde{\sigma})^{\circ})$$

Then we see that

$$\operatorname{grad} \rho^{\widetilde{\sigma}} = X \text{ and } \rho \text{ is downward convex.}$$

Also we see that

$$\rho^{\widetilde{\sigma}}(w) \to \infty \quad (w \to \partial\widetilde{\sigma})$$

$$\rho^{\widetilde{\sigma}}(tw) \to -\infty \quad (t \to \infty) \text{ for each } w \in (\widetilde{\sigma})^{\circ}$$

The outline of the proof of Theorems 22.1 and 22.2

Hence we see that

$\rho^{\widetilde{\sigma}}$ **has no minimal point.**

Furthermore, we can show that

the integral curve $\xi(t)$ **of** $X^{\widetilde{\sigma}}$ **starting at** w_0

converges to a pt. w_1 **of** $\partial\widetilde{\sigma}$ **in finite time** T**.**

Since $F_t = \eta_{(\pi\circ\phi)_*(\widetilde{\xi(t)})}(M)$,

F_t **collapses to the lower dim. focal submanifold**

$\eta_{(\pi\circ\phi)_*(\widetilde{w}_1)}(M)$ **in the time** T**.**

q.e.d.