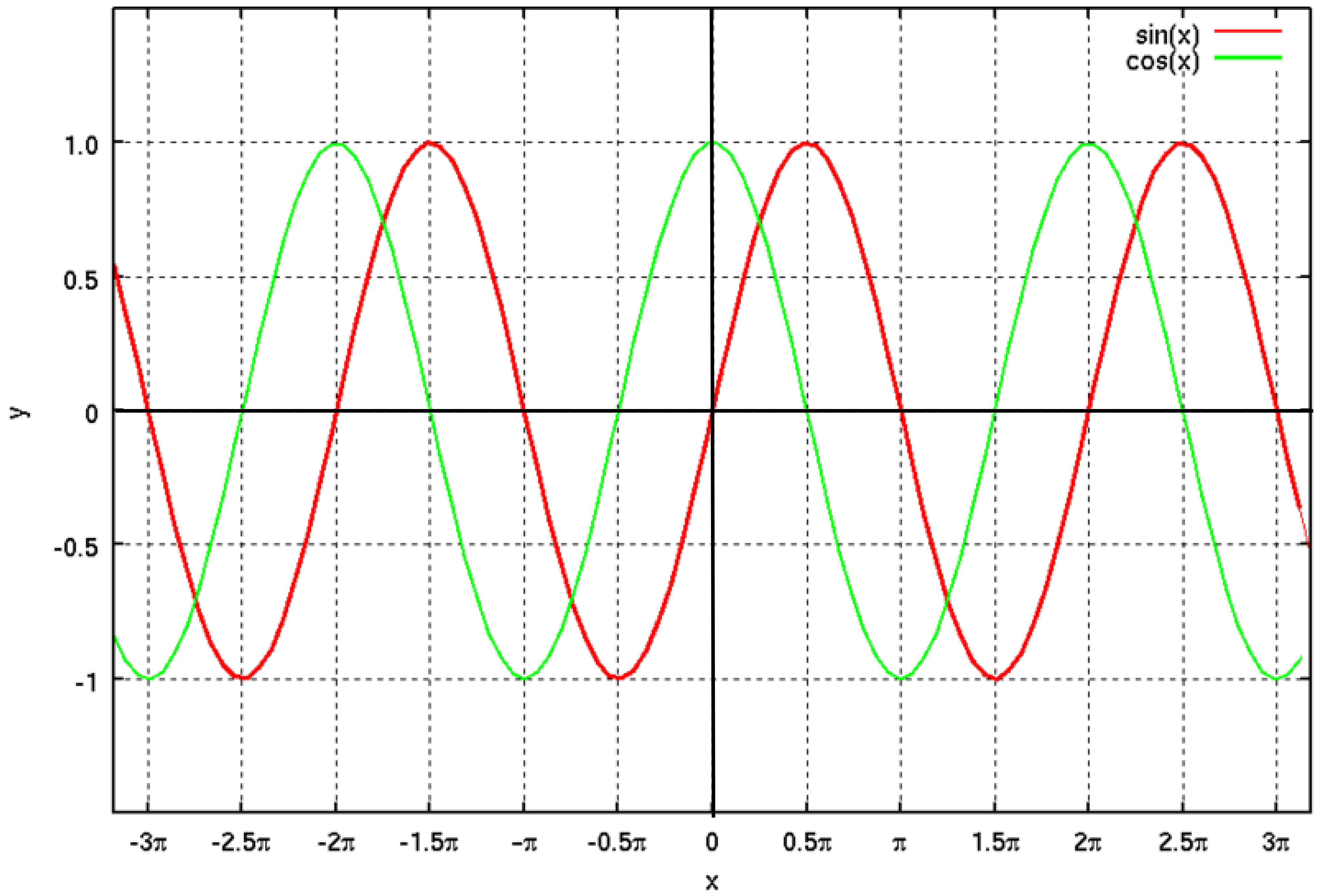
三角関数（＝円関数）とフーリエ級数（変換）

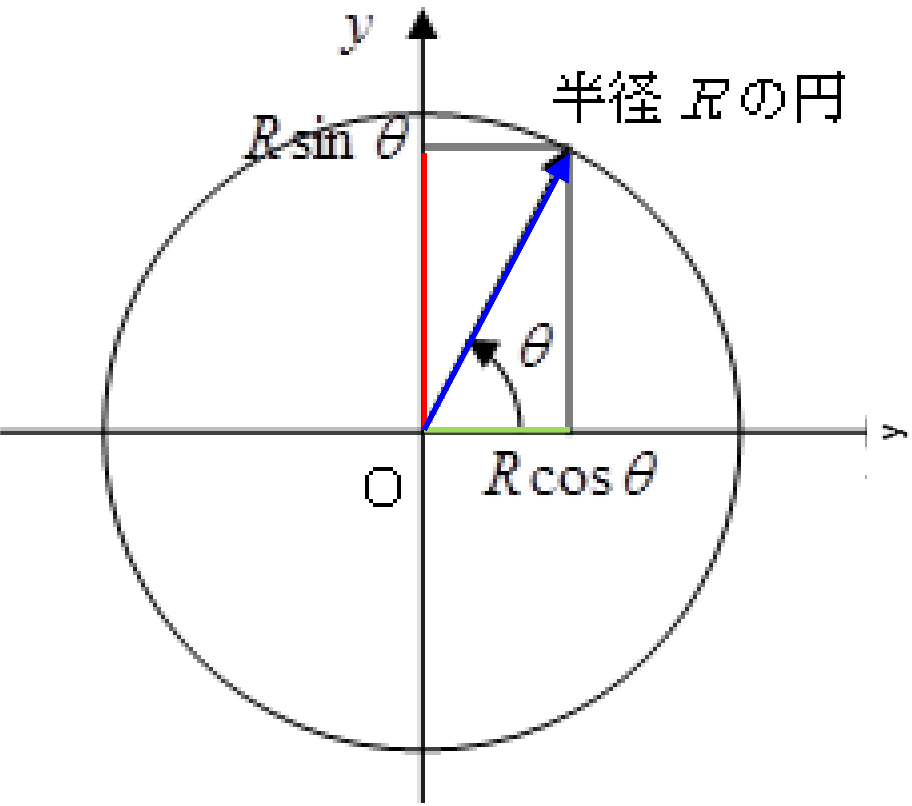
$$\sin x$$

$$\cos x$$

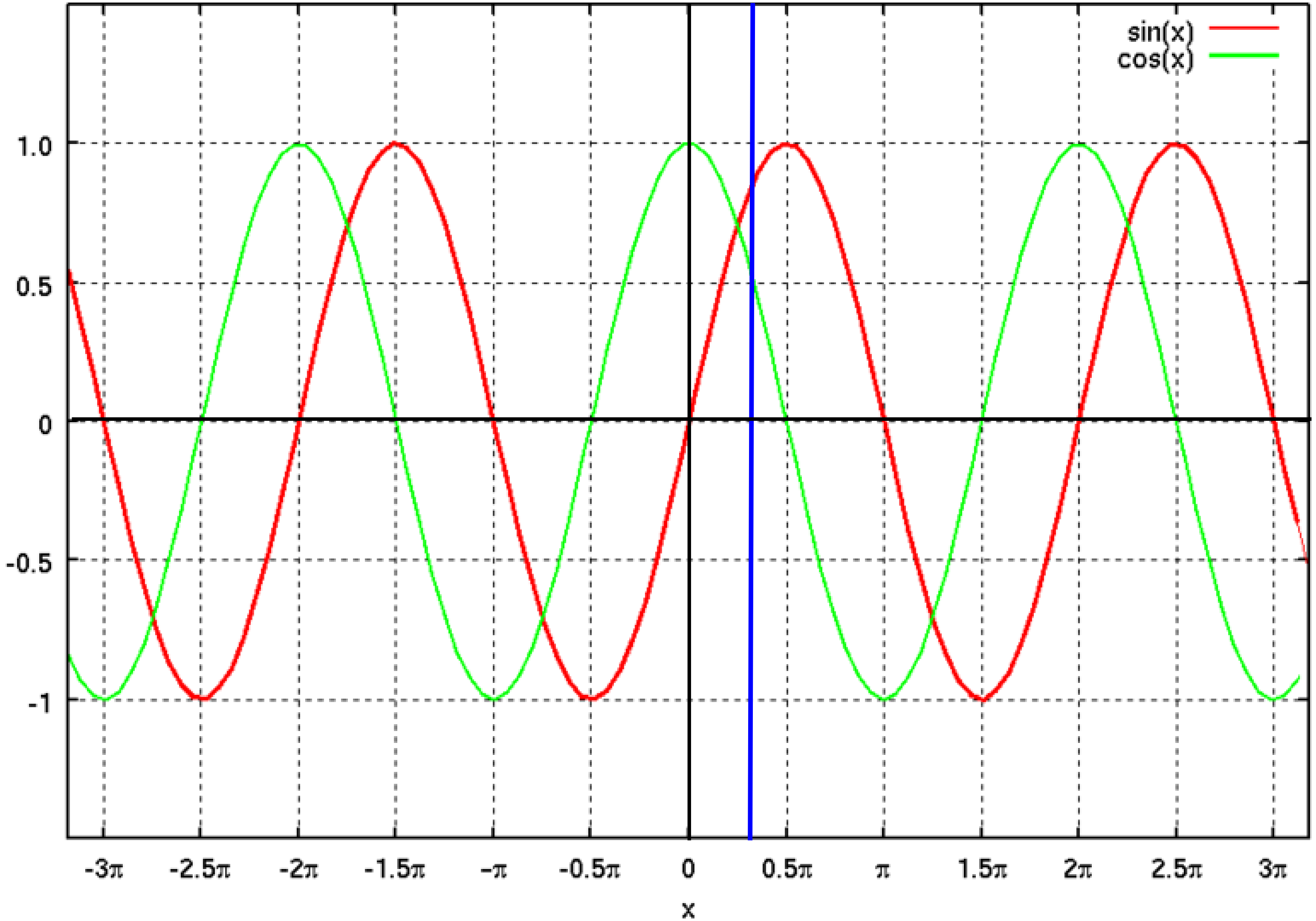


π＝180°

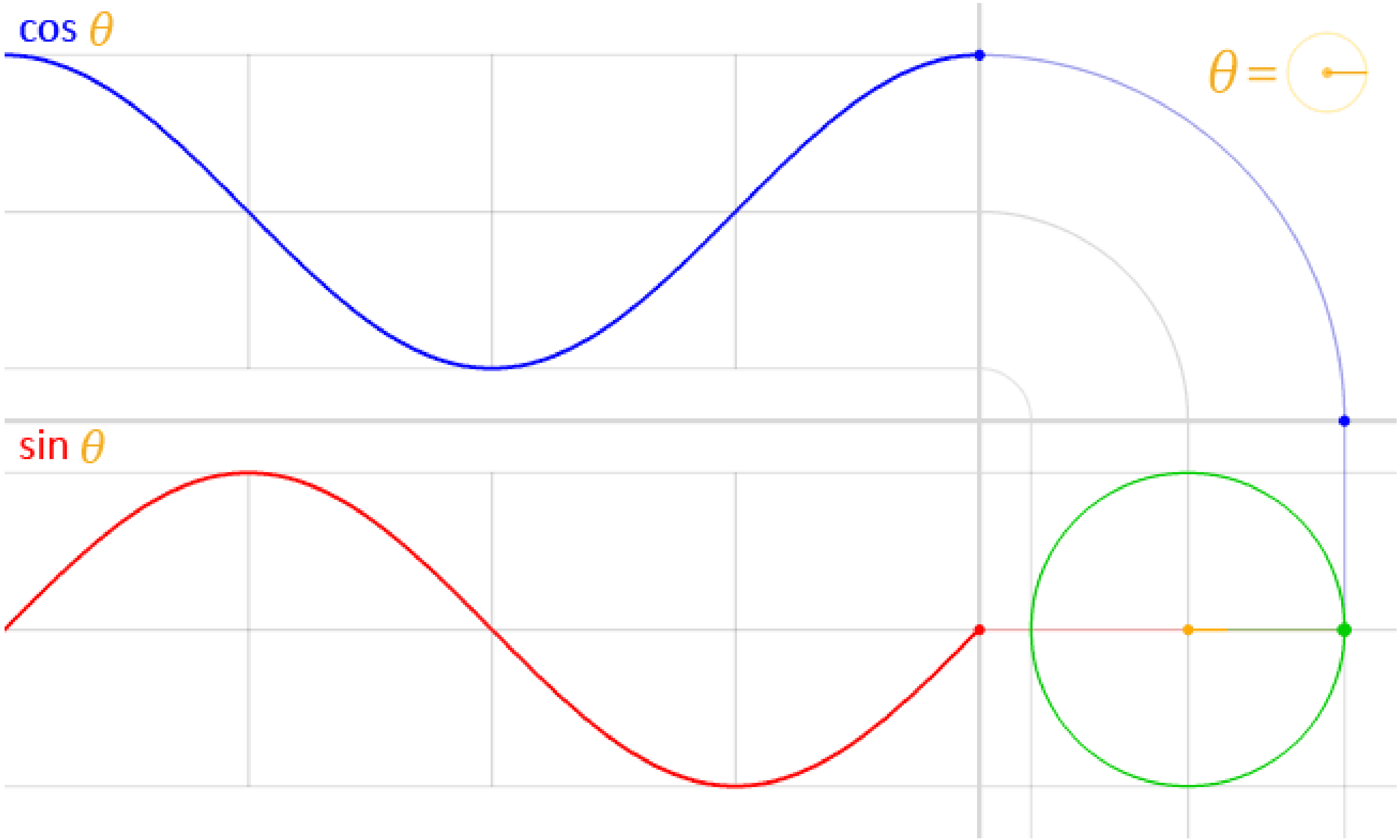
R=1　横軸 x= θ



y
半径 R の円
R sin θ
θ
O
R cos θ
y



sin(x)
cos(x)
1.0
0.5
0
-0.5
-1
-3π
-2.5π
-2π
-1.5π
-π
-0.5π
0
0.5π
π
1.5π
2π
2.5π
3π
x



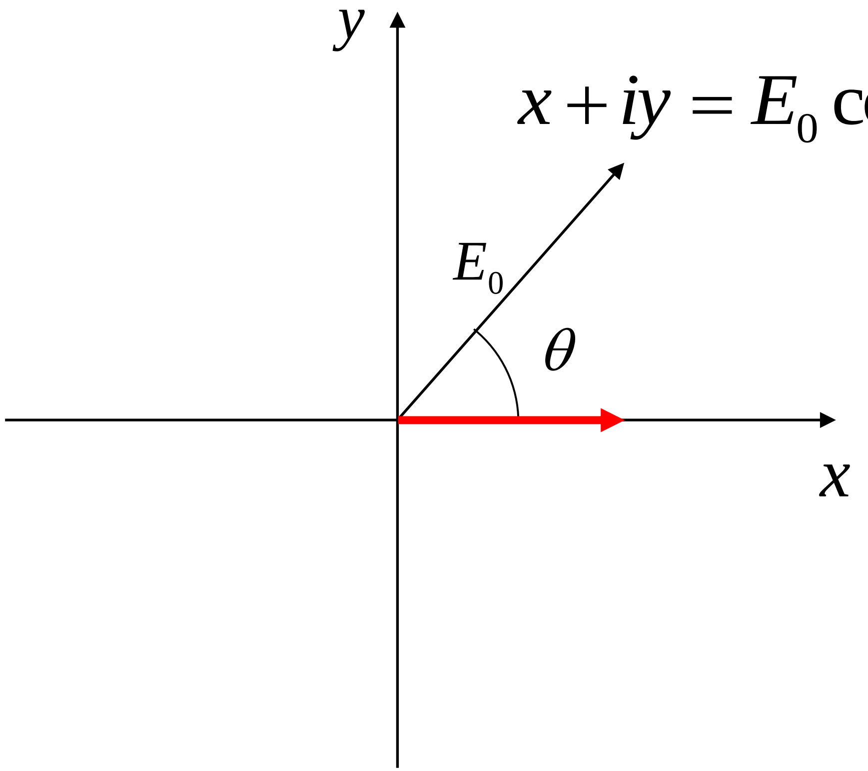
cos θ
$\theta =$
sin θ

波の振幅の複素表示

$$E = E_0 e^{i\theta} = E_0 e^{i\omega t}$$

観測されるのはこの実部

$$\mathrm{Re}\, E = E_0 \cos \omega t$$



$$x + iy = E_0 \cos\theta + iE_0 \sin\theta$$

位相θ

オイラーの公式

$$e^{i\pi} = -1 \quad e = 2.71828\cdots$$

自然対数の底

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$E_0 e^{i\omega t} = E_0 \cos\omega t + iE_0 \sin\omega t$$

ω, t, k, xで波を表す

t 時間

x 距離

T 周期

λ 波長

$$\sin(\omega t - kx)$$

$$\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right)$$

角周波数$\omega = \frac{2\pi}{T}$　波数$k = \frac{2\pi}{\lambda}$

可視光の波数

$$\lambda = 500\text{nm}$$

$$\frac{k}{2\pi} = \frac{1}{\lambda}$$

$$= \frac{1}{500\text{nm}}$$

$$= \frac{1}{500 \times 10^{-9}\text{m}}$$

$$= \frac{1}{500 \times 10^{-7}\text{cm}}$$

$$= 2 \times 10^{4}\text{cm}^{-1}$$

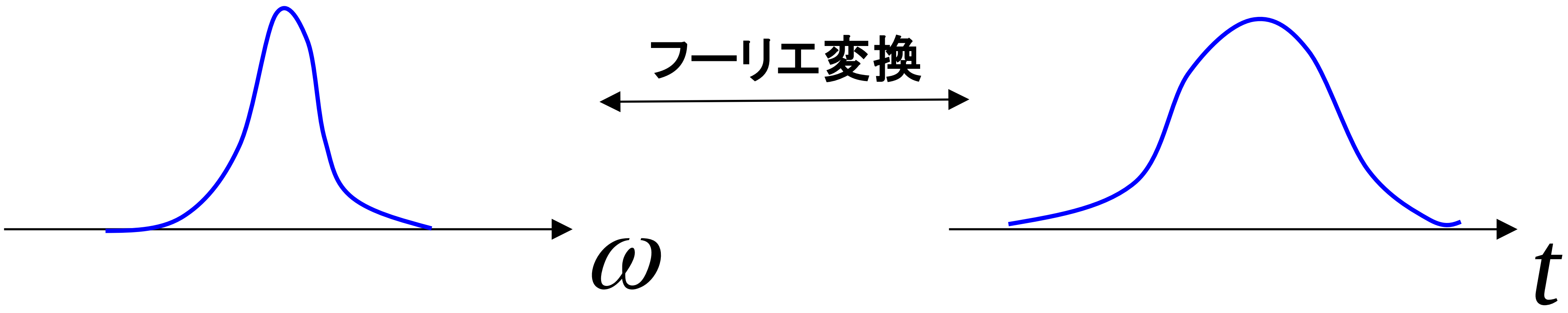
1cmあたり20000個の波を持つ（20000波長）

フーリエ変換とは

波の ω 依存性
と
波の t 依存性

波の k 依存性
と
波の x 依存性

の関係を与える変換式



フーリエ変換

$$F(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

ω $\quad t \quad\quad t \quad \omega$

$f(t)=e^{i\omega_0 t}$ $\quad F(\omega)=2\pi\delta(\omega-\omega_0)$ $\quad \delta(x)$:デルタ関数

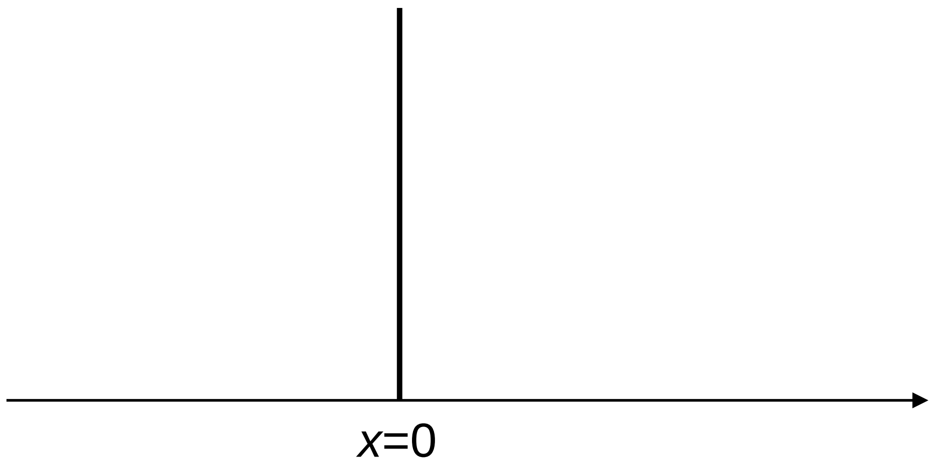
$$F(\omega)=\int_{-\infty}^{\infty} e^{i\omega_0 t}e^{-i\omega t}dt = 2\pi\delta(\omega-\omega_0) \qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{i\omega t}d\omega=e^{i\omega_0 t}$$

デルタ関数

$$\delta(x) = \begin{cases} 0 & (x \neq 0) \\ \infty & (x = 0) \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$



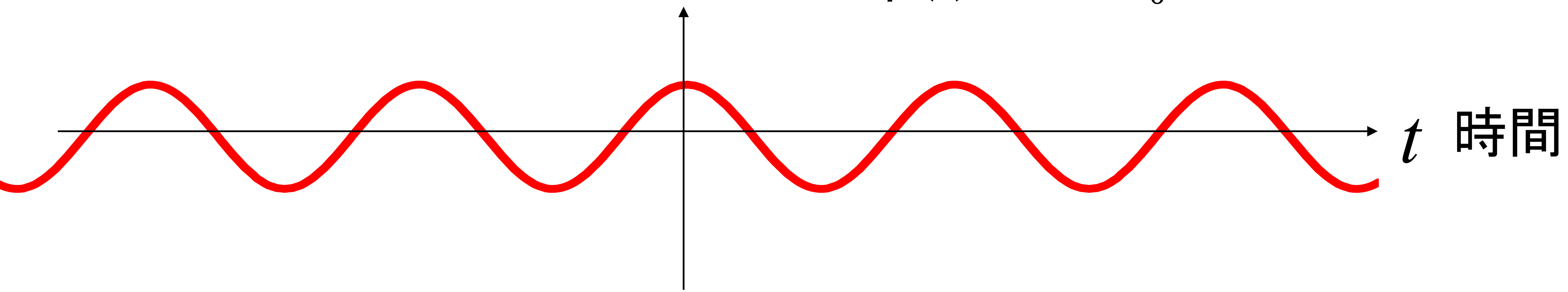
cos関数、sin関数のフーリエ変換は？

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \sin\theta = -\frac{i}{2}(e^{i\theta} - e^{-i\theta})$$ であることより

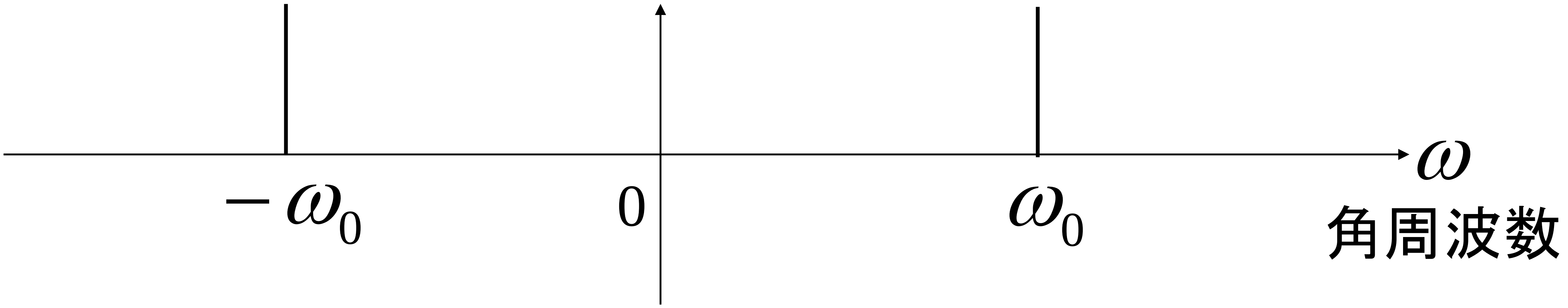
波の伝播

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$f(t) = \cos\omega_0 t$$



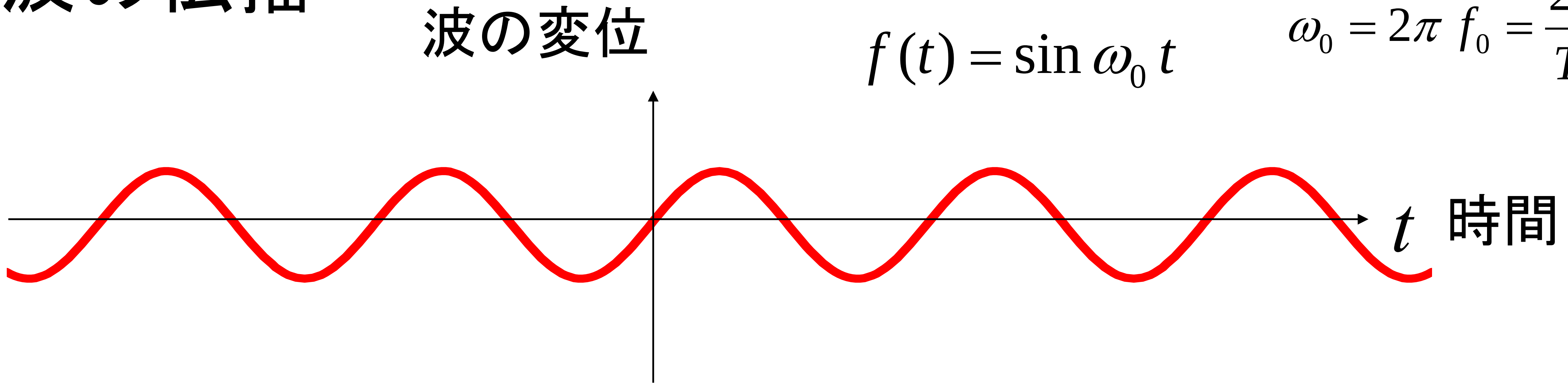
$$F(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



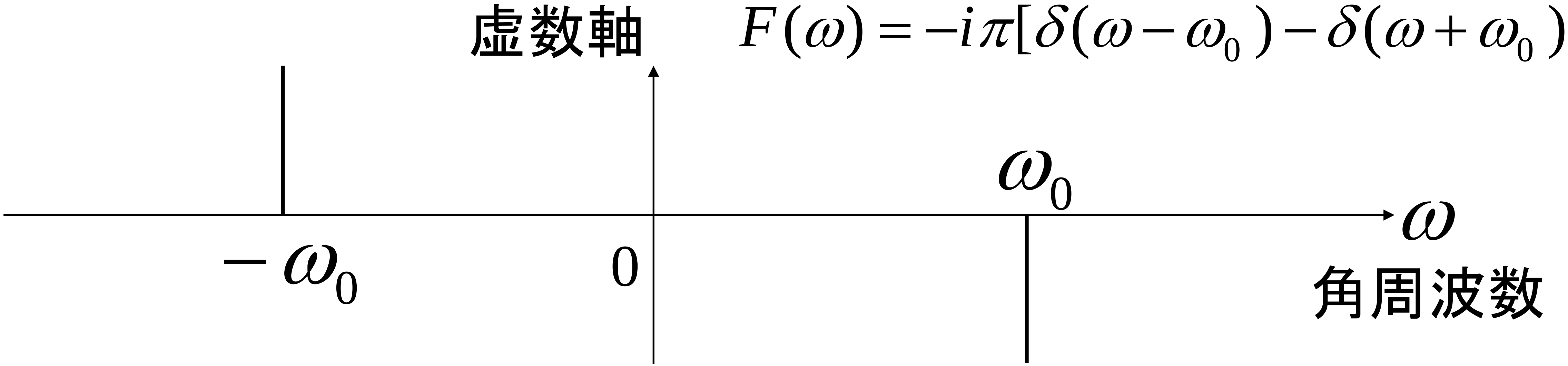
波の伝播

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

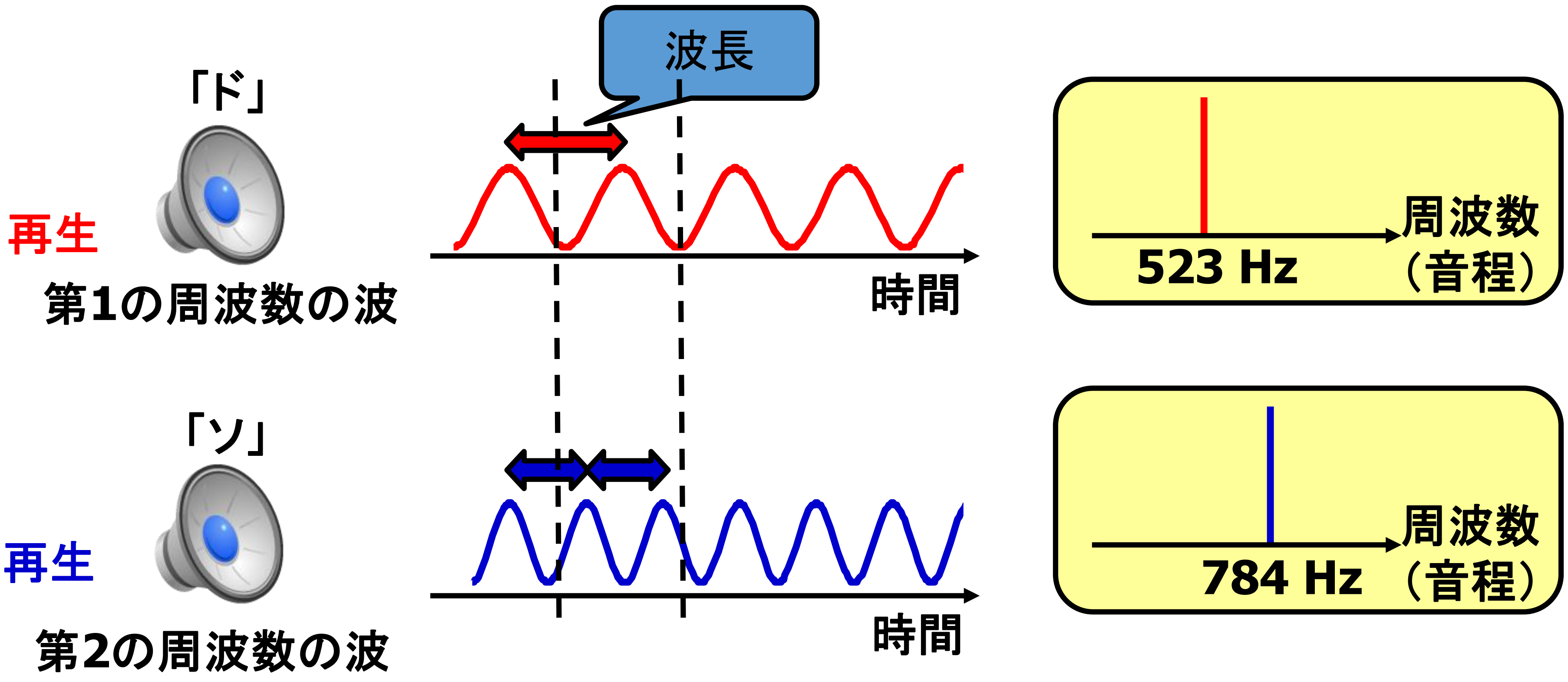
$$f(t) = \sin \omega_0 t$$



$$F(\omega) = -i\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

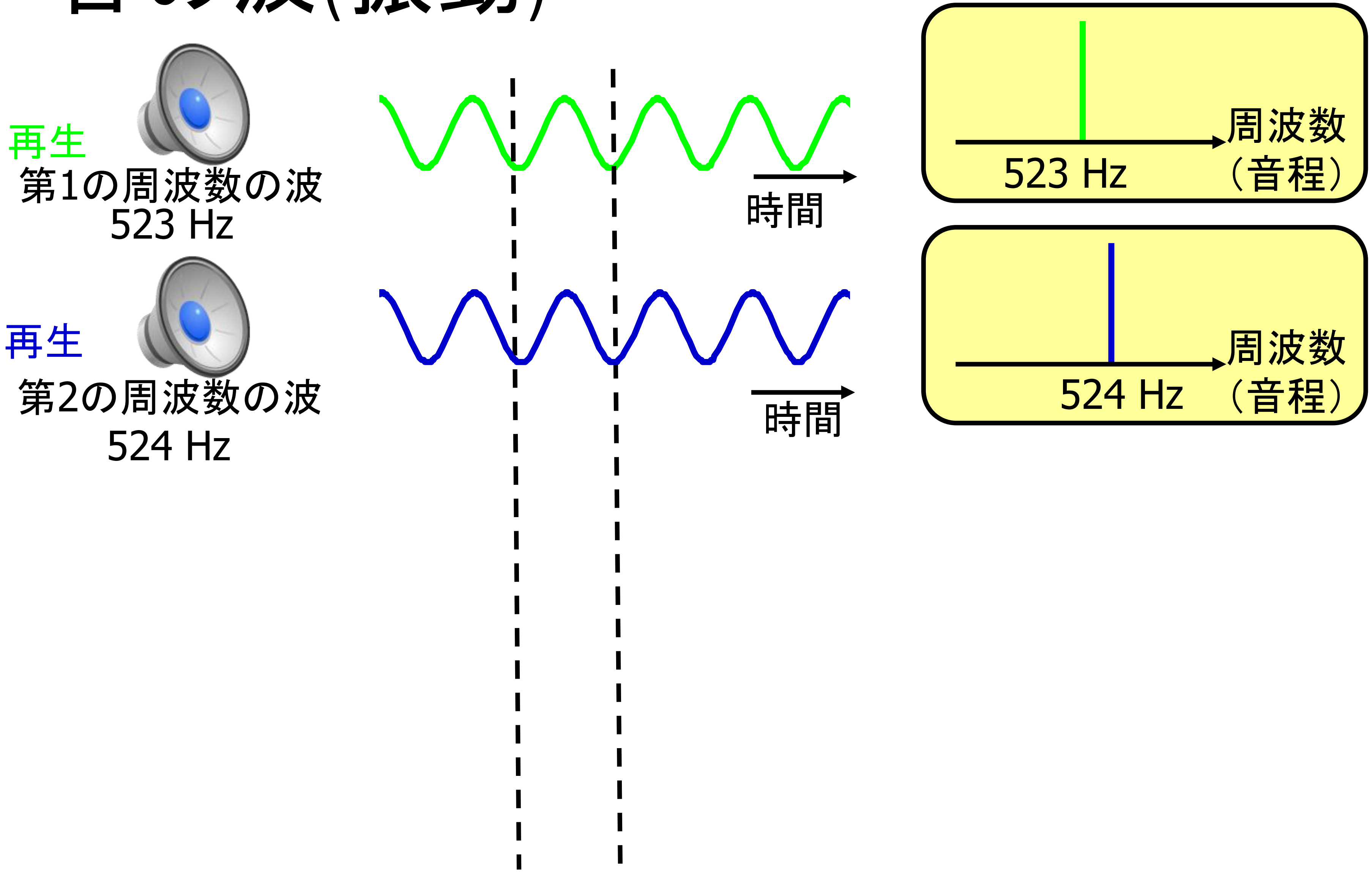


音の波(振動)



by Minoshima

音の波(振動)



by Minoshima

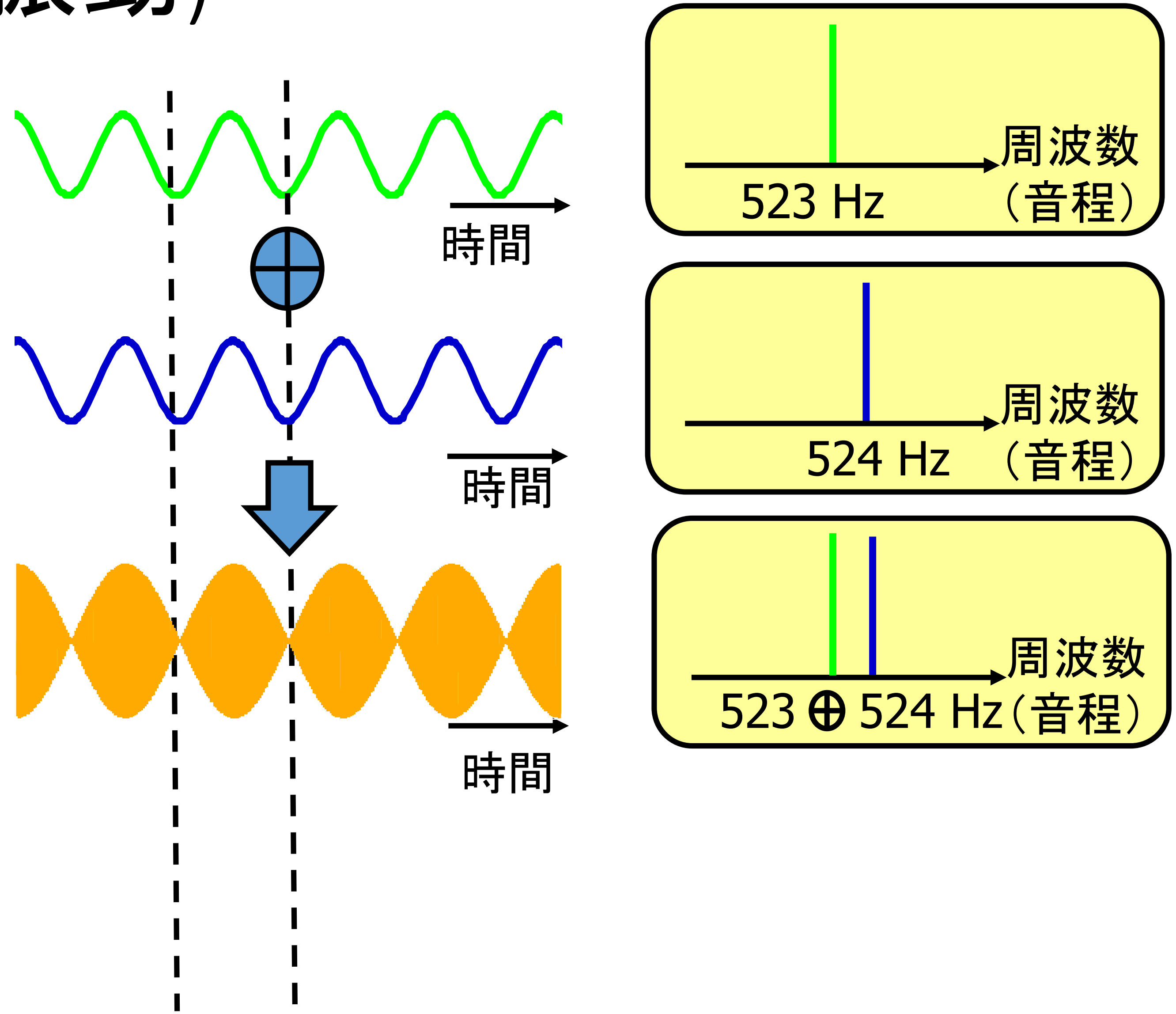
音の波(振動)

第1の周波数の波
523 Hz

第2の周波数の波
524 Hz

再生

2つの重ね合わせ
523 ⊕ 524 Hz



by Minoshima

音の波(振動)

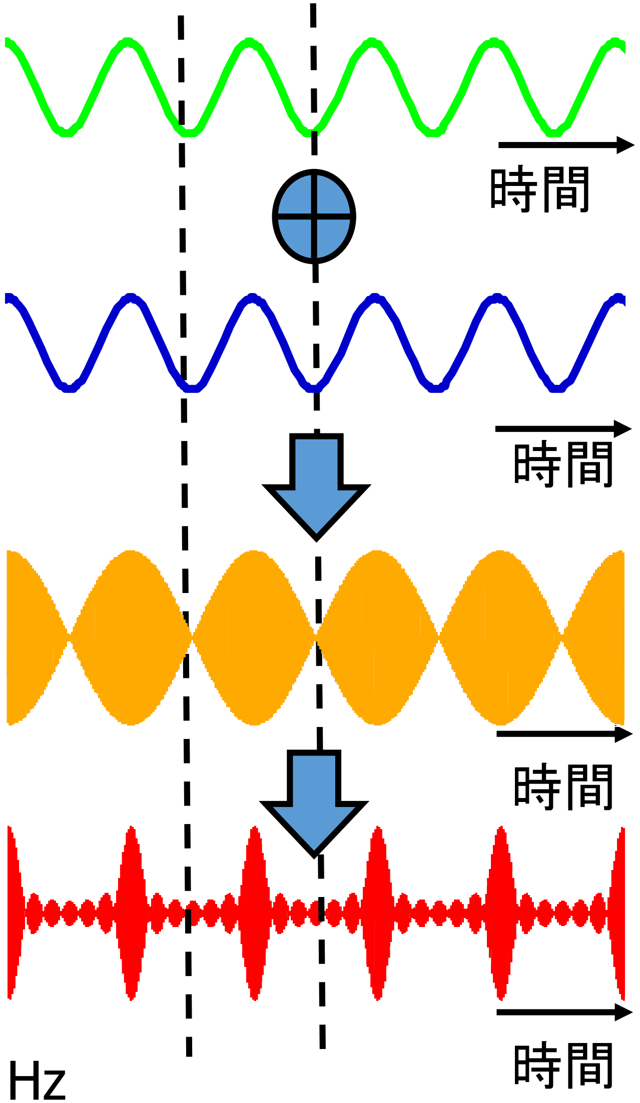
第1の周波数の波
523 Hz

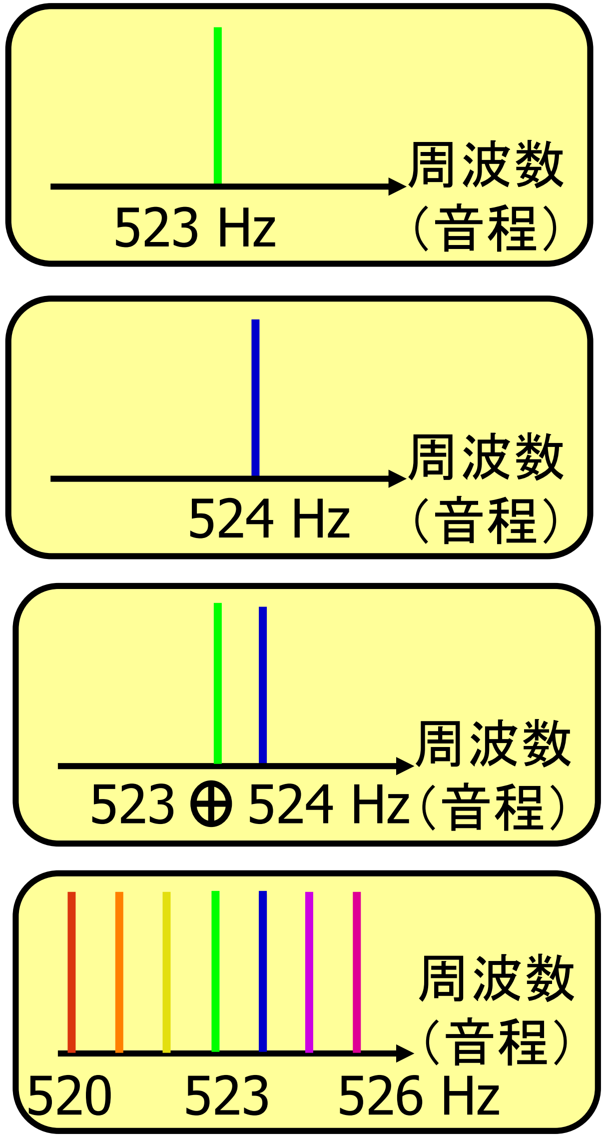
第2の周波数の波
524 Hz

2つの重ね合わせ
523 ⊕ 524 Hz

再生

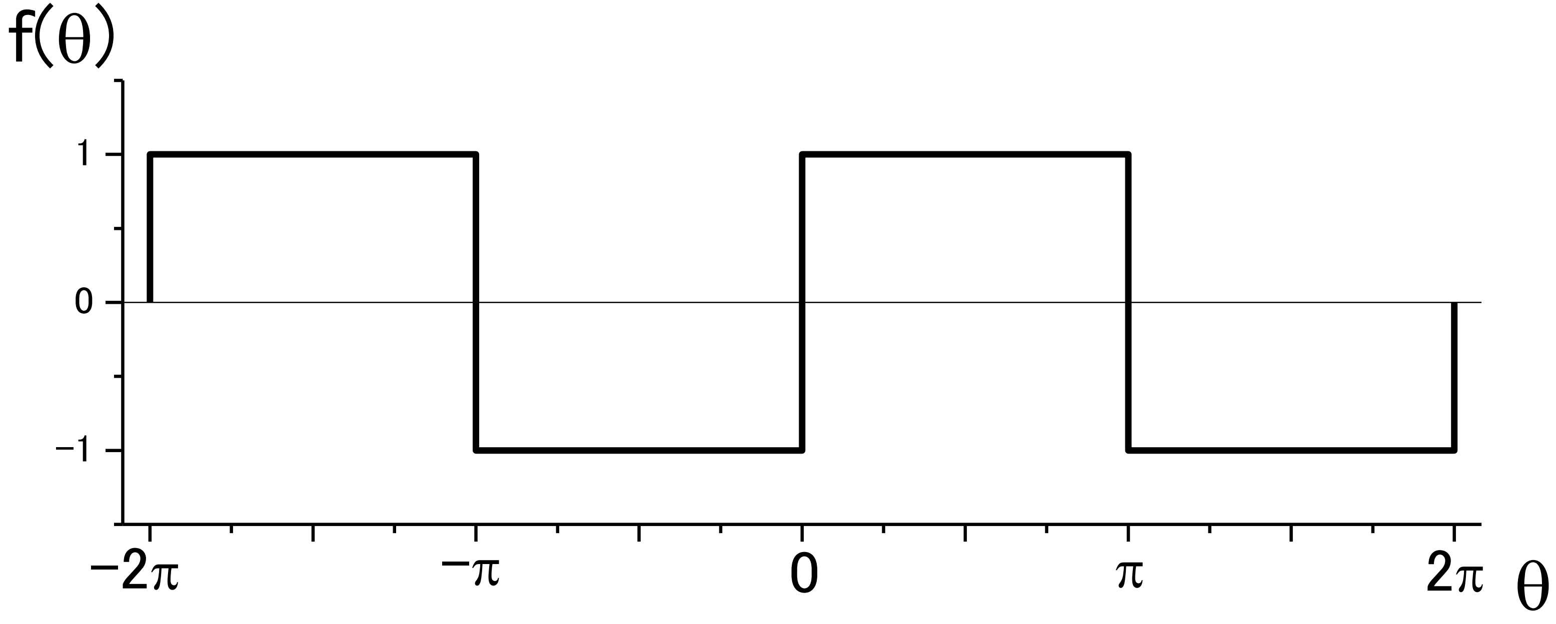
7つの重ね合わせ
520⊕. . .523. .⊕ 526 Hz

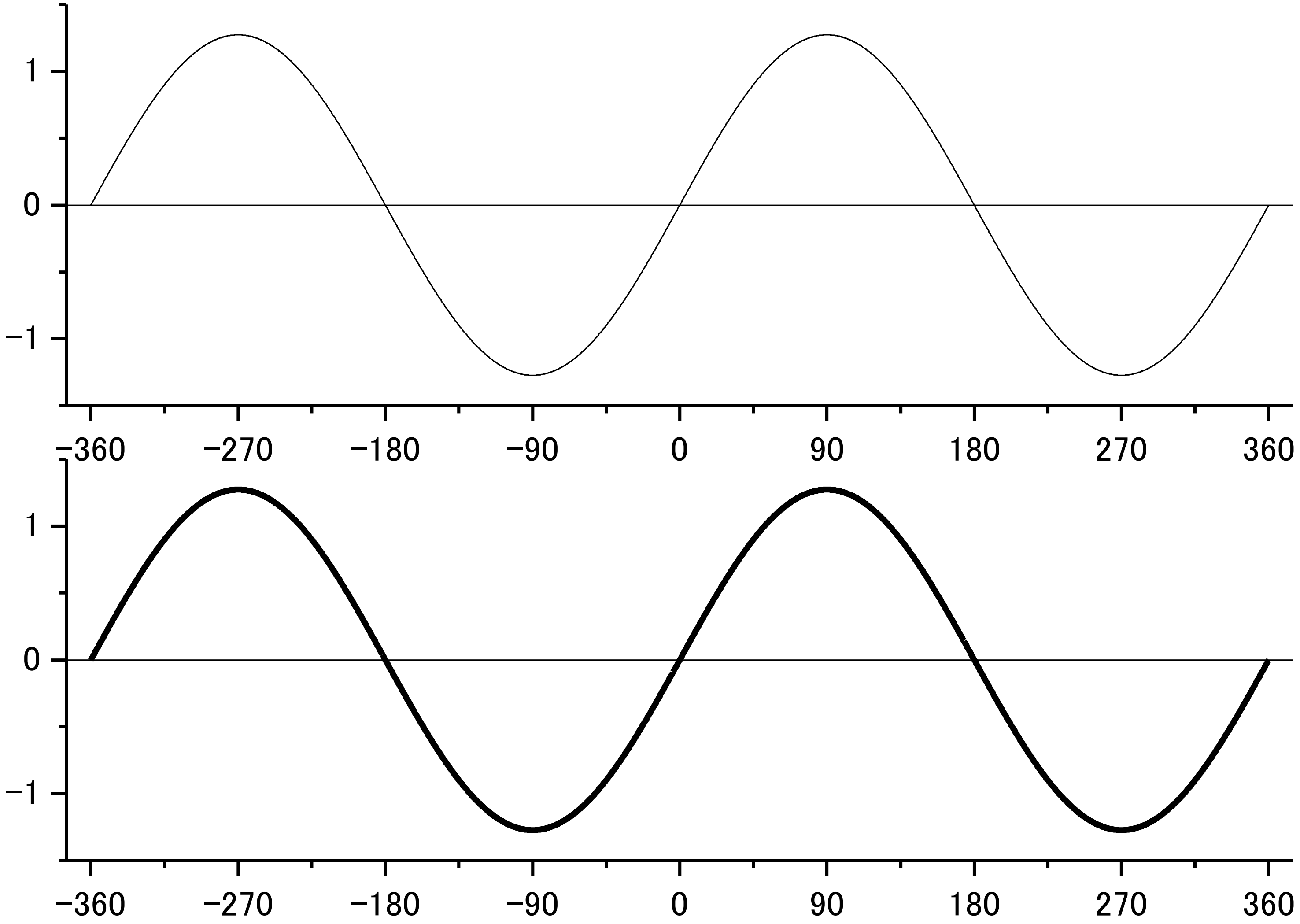




by Minoshima

以下の関数のフーリエ変換は？

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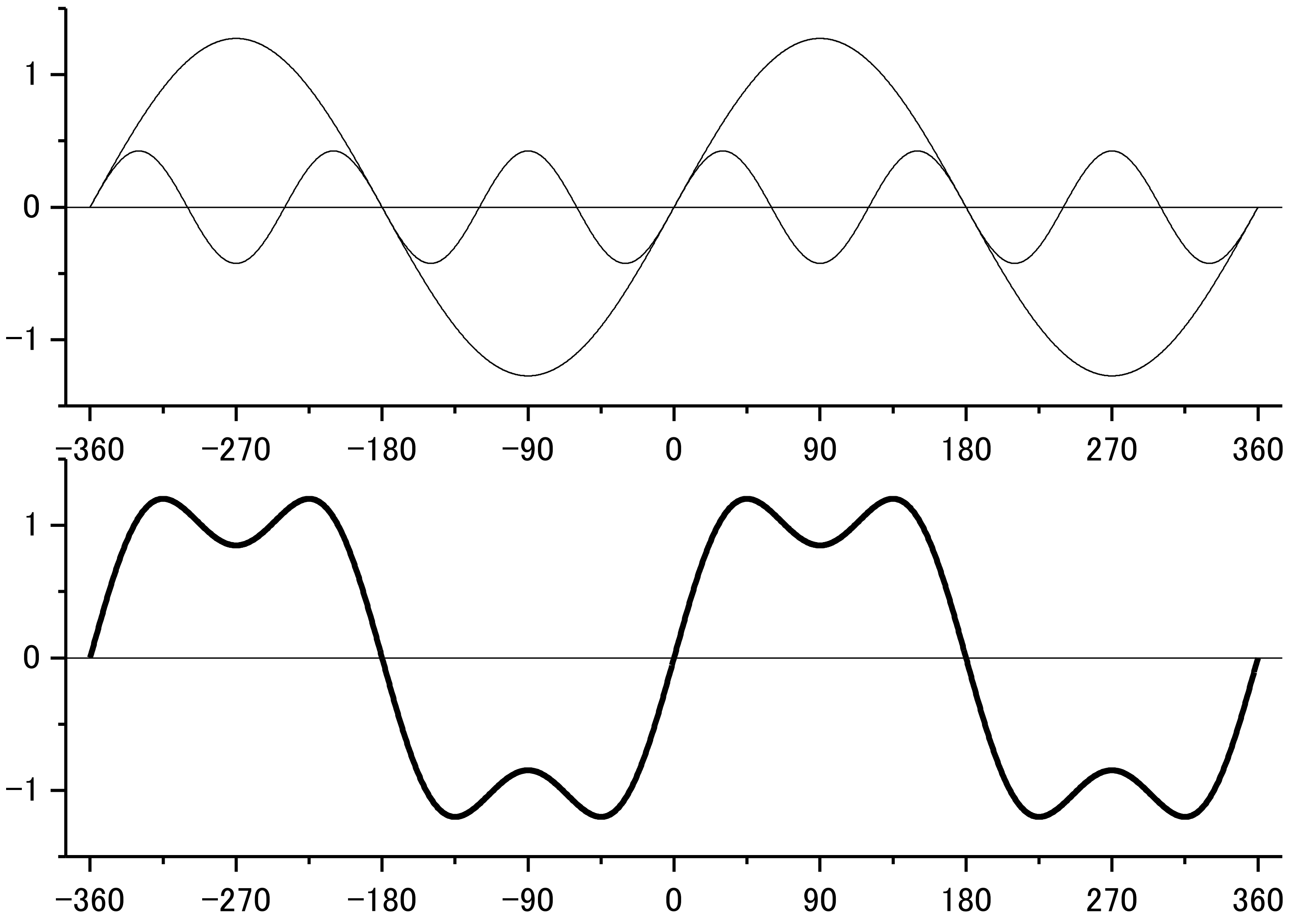
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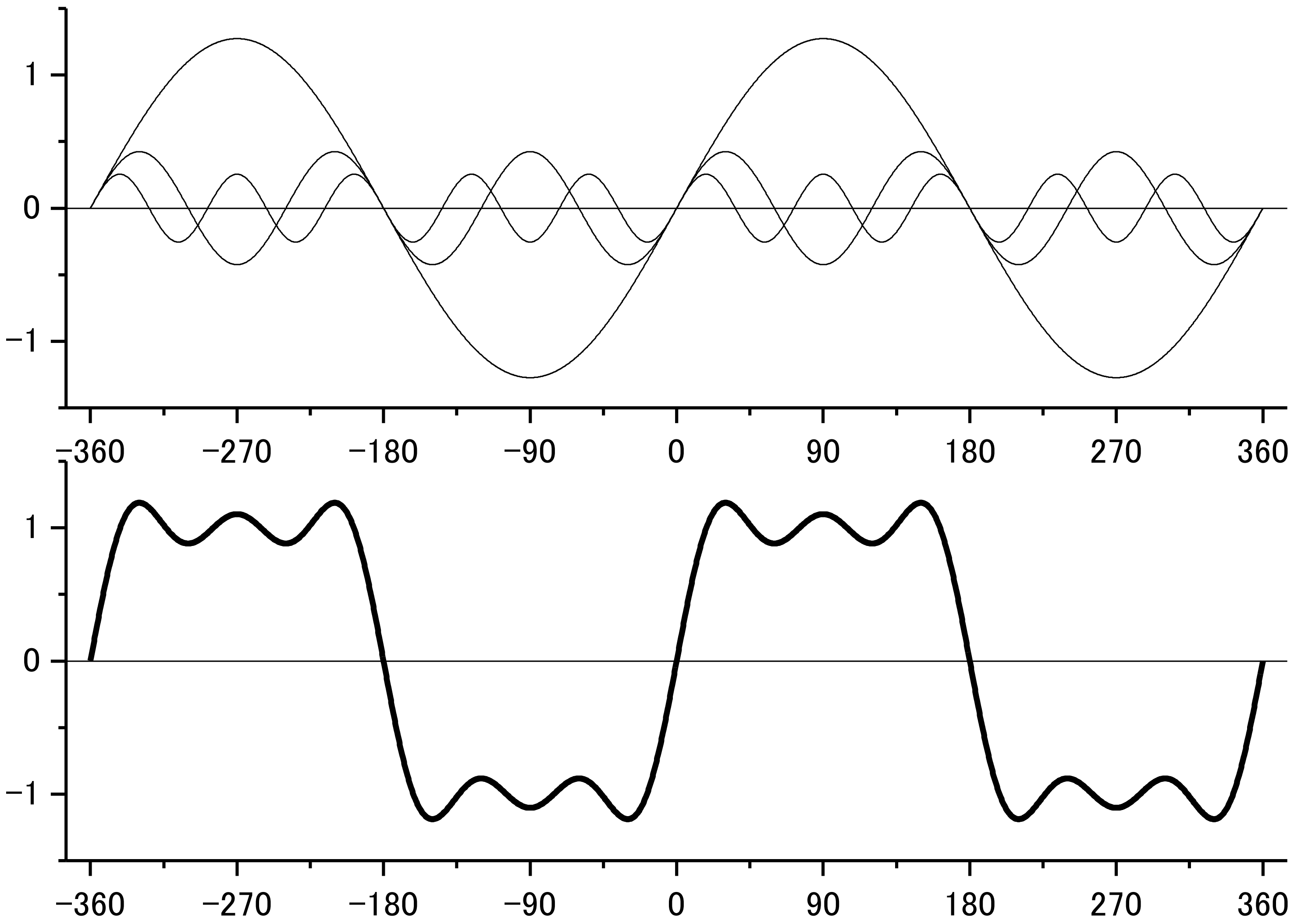
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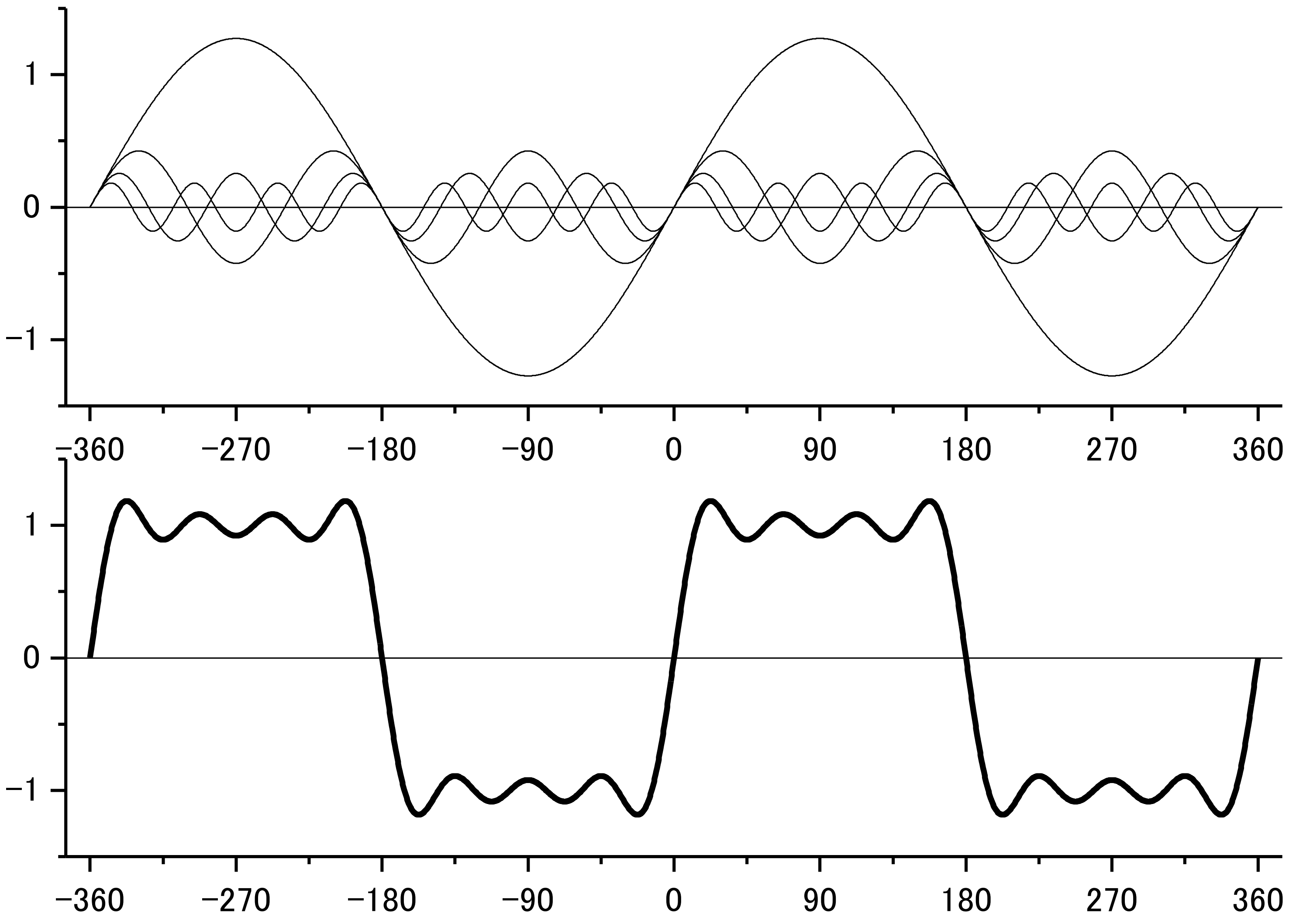

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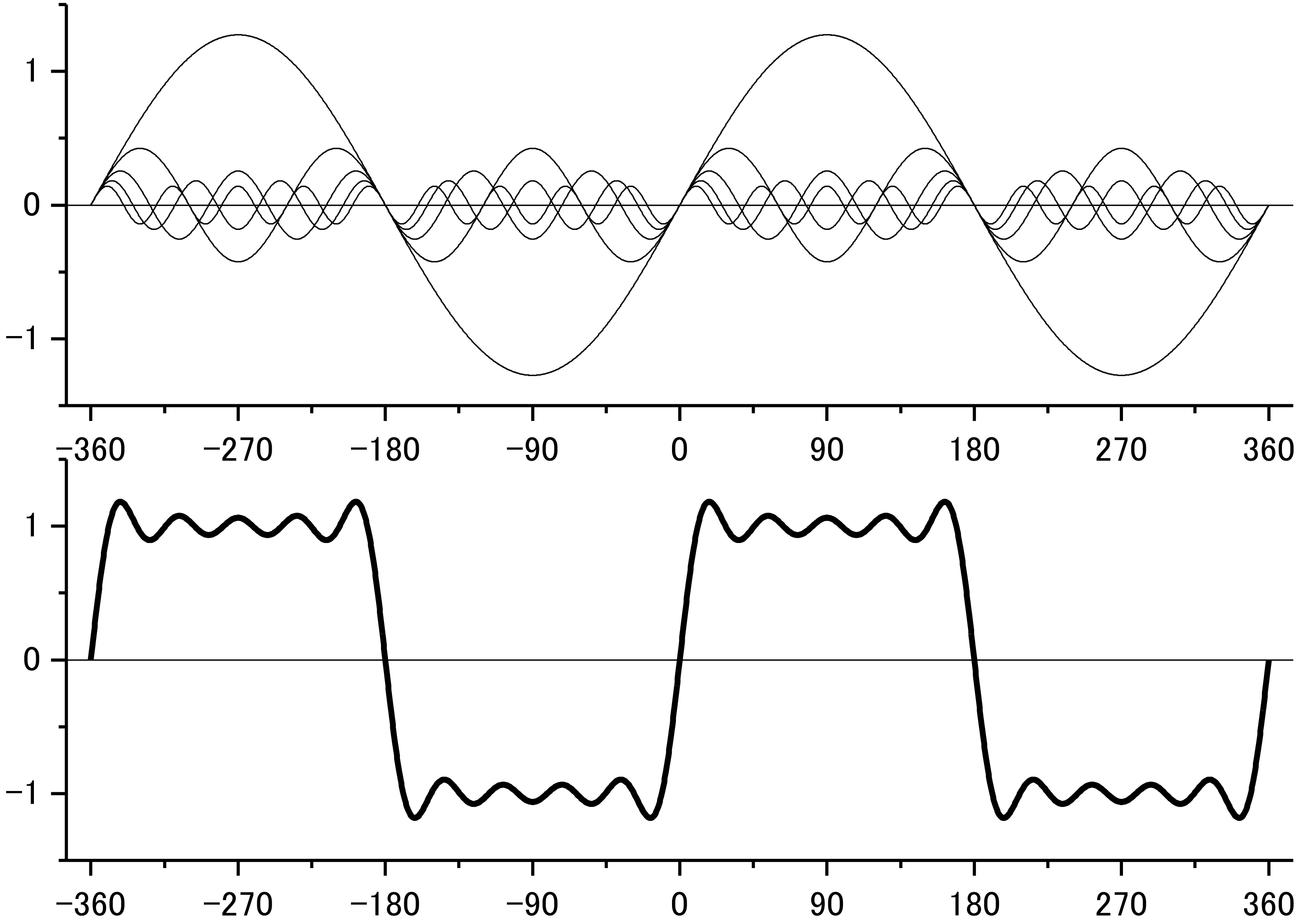
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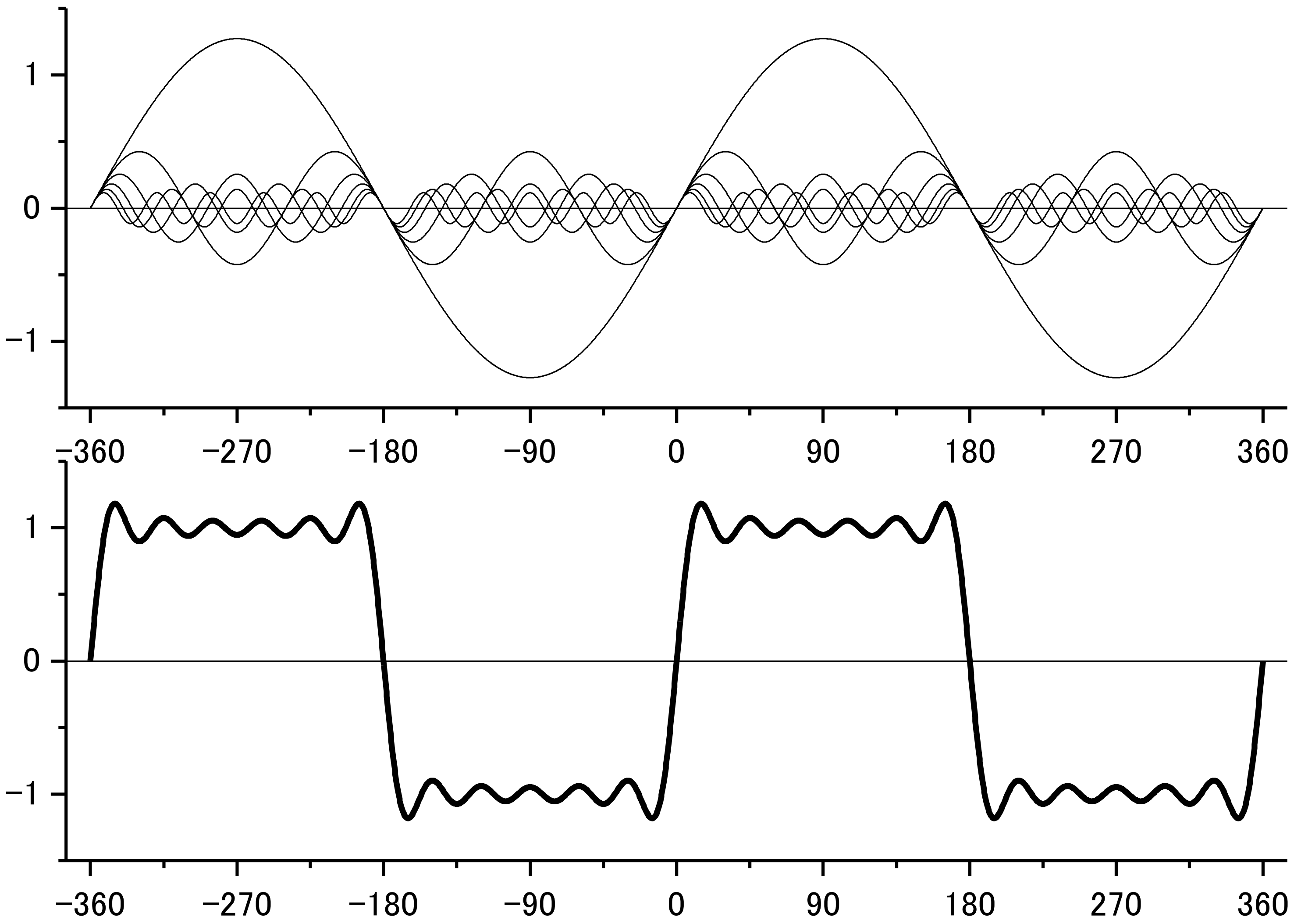
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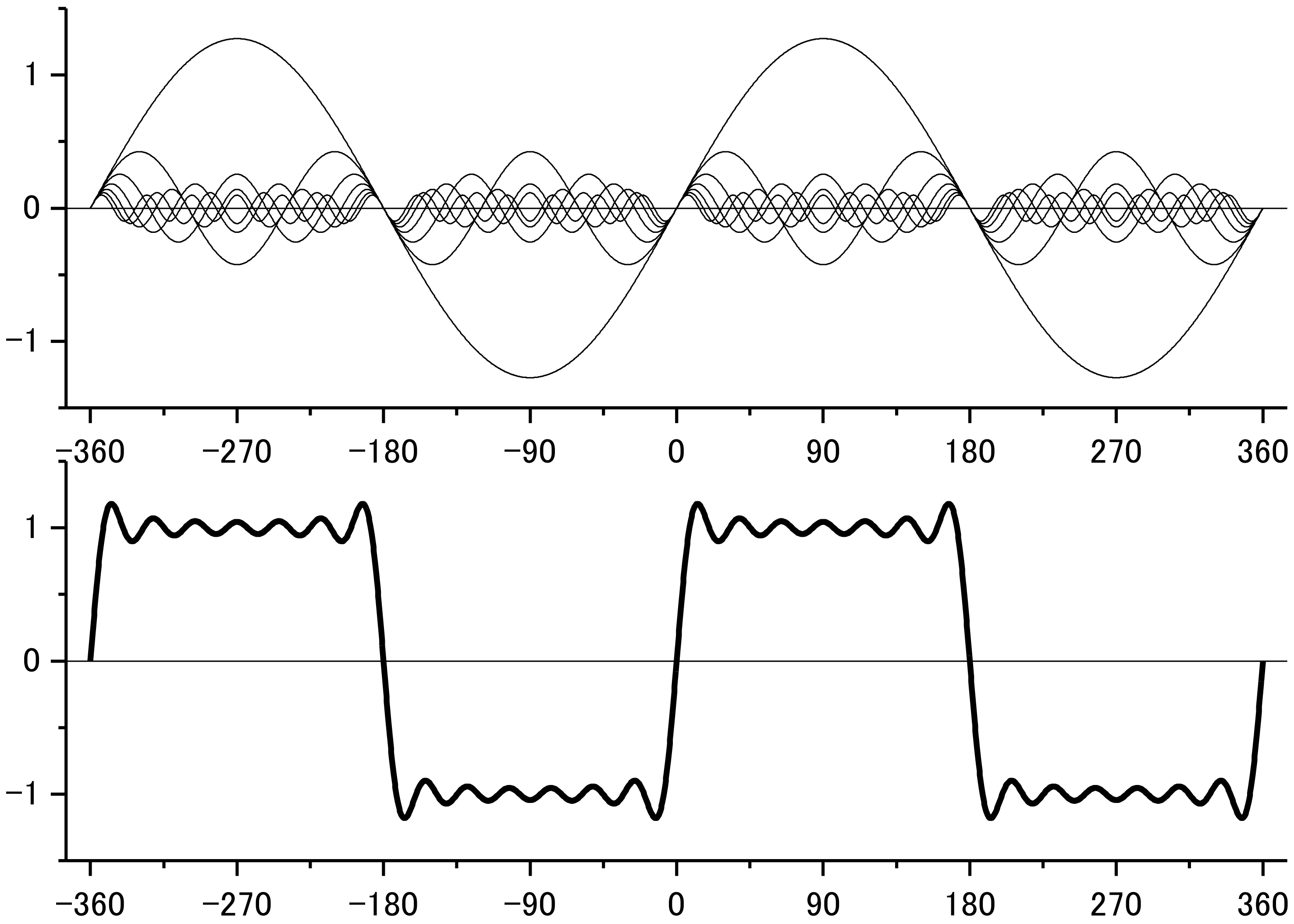

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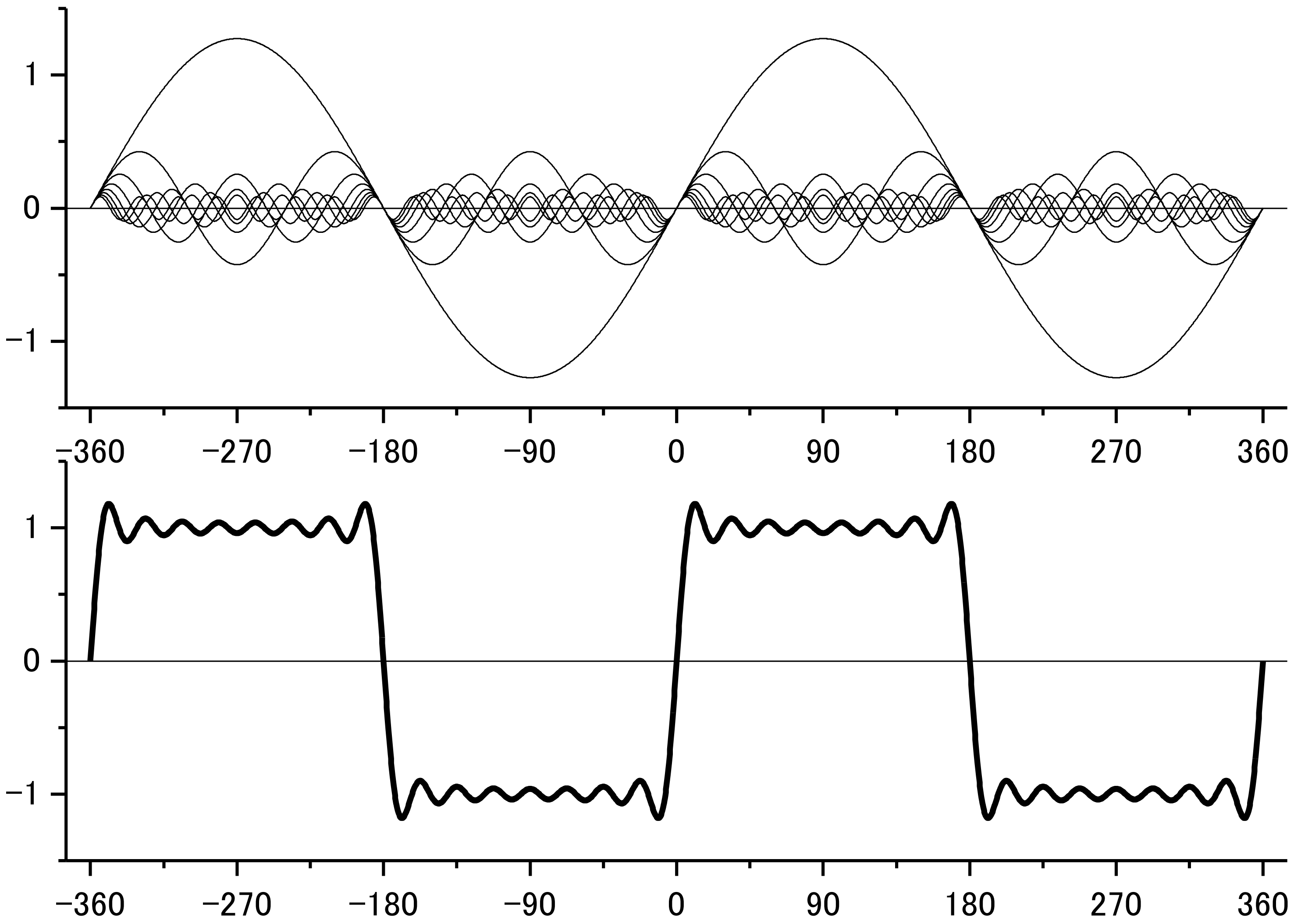
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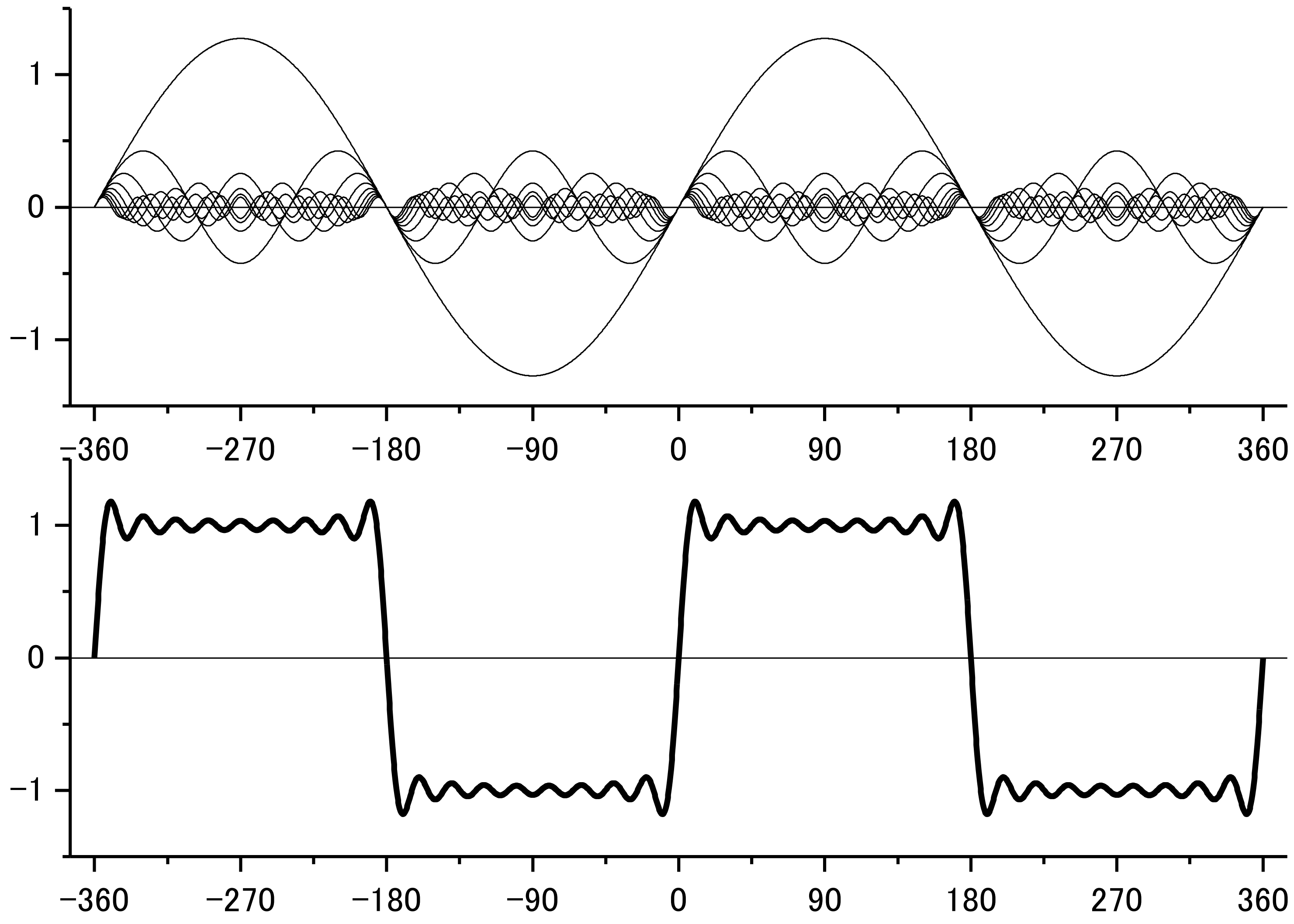

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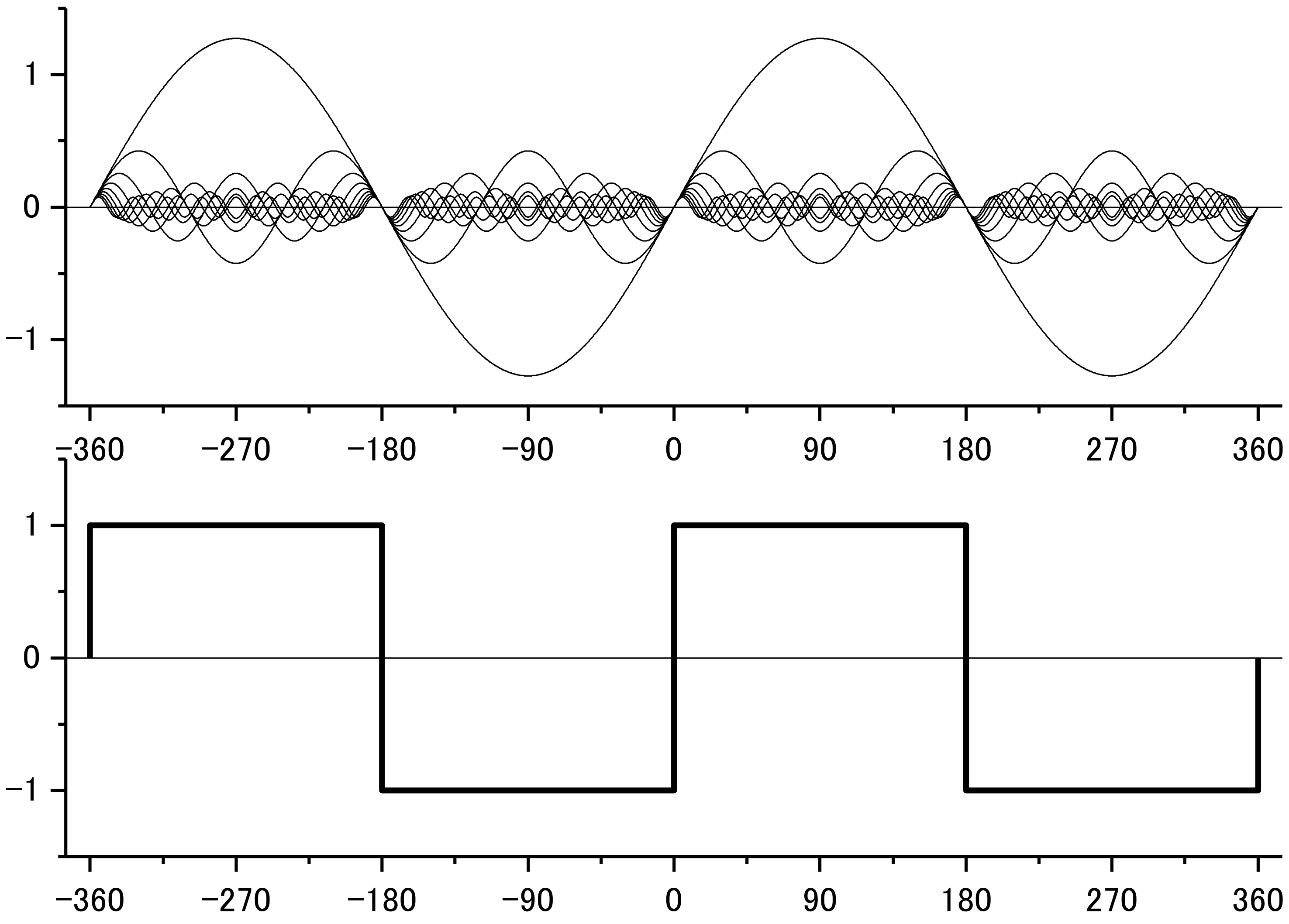
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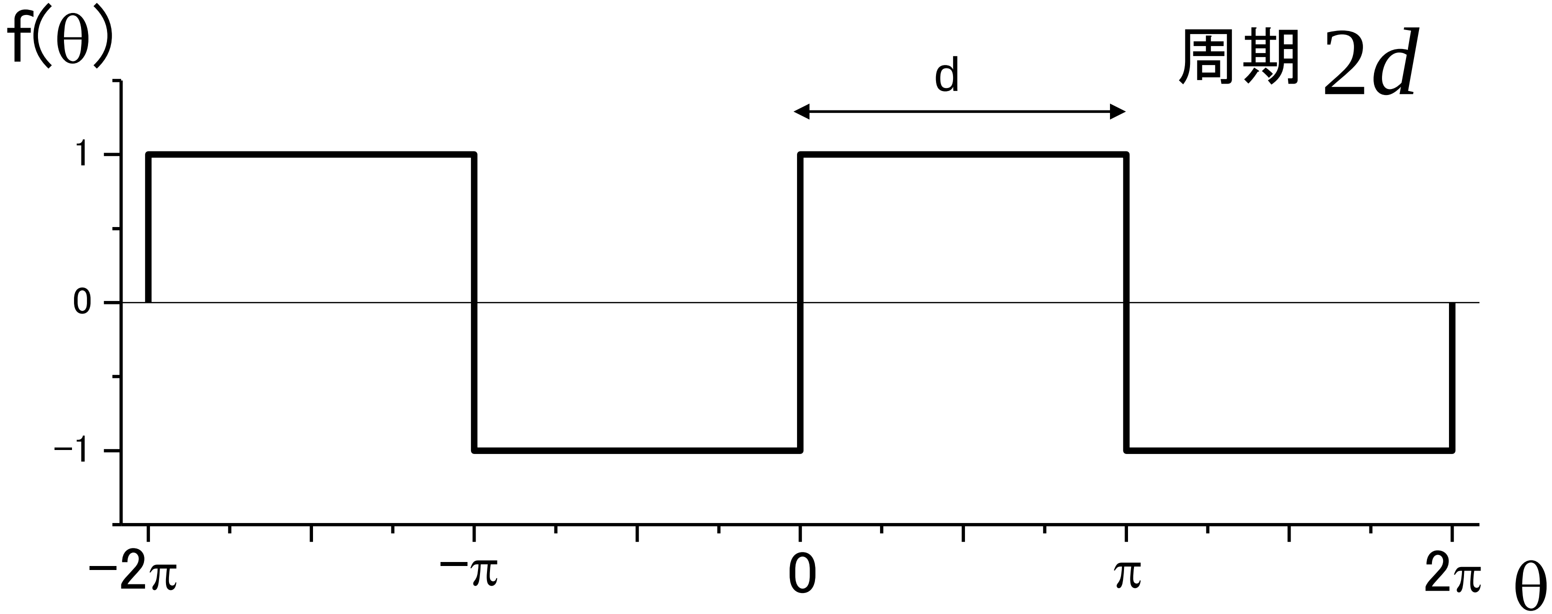
フーリエ級数

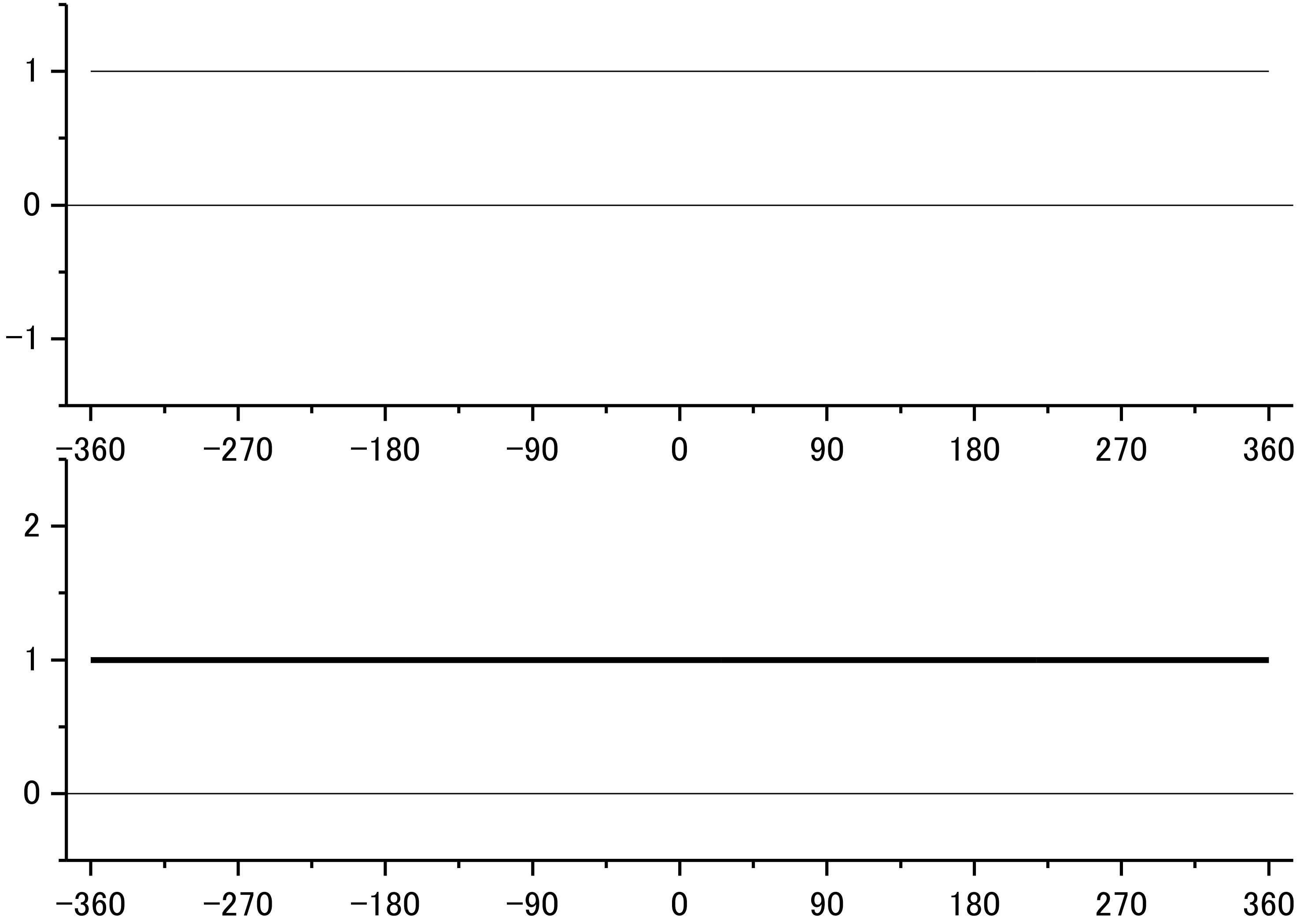


周期 $2d$

$$f(\theta)=\frac{4}{\pi}\left(\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\frac{1}{7}\sin 7\theta+\cdots\right)$$

$$f(\theta=\frac{\pi}{d}x)=\frac{4}{\pi}\left(\sin\frac{\pi}{d}x+\frac{1}{3}\sin 3\frac{\pi}{d}x+\frac{1}{5}\sin 5\frac{\pi}{d}x+\frac{1}{7}\sin 7\frac{\pi}{d}x+\cdots\right)$$

矩形関数は高い空間周波数成分 $k_n=n\dfrac{\pi}{d}$ を含む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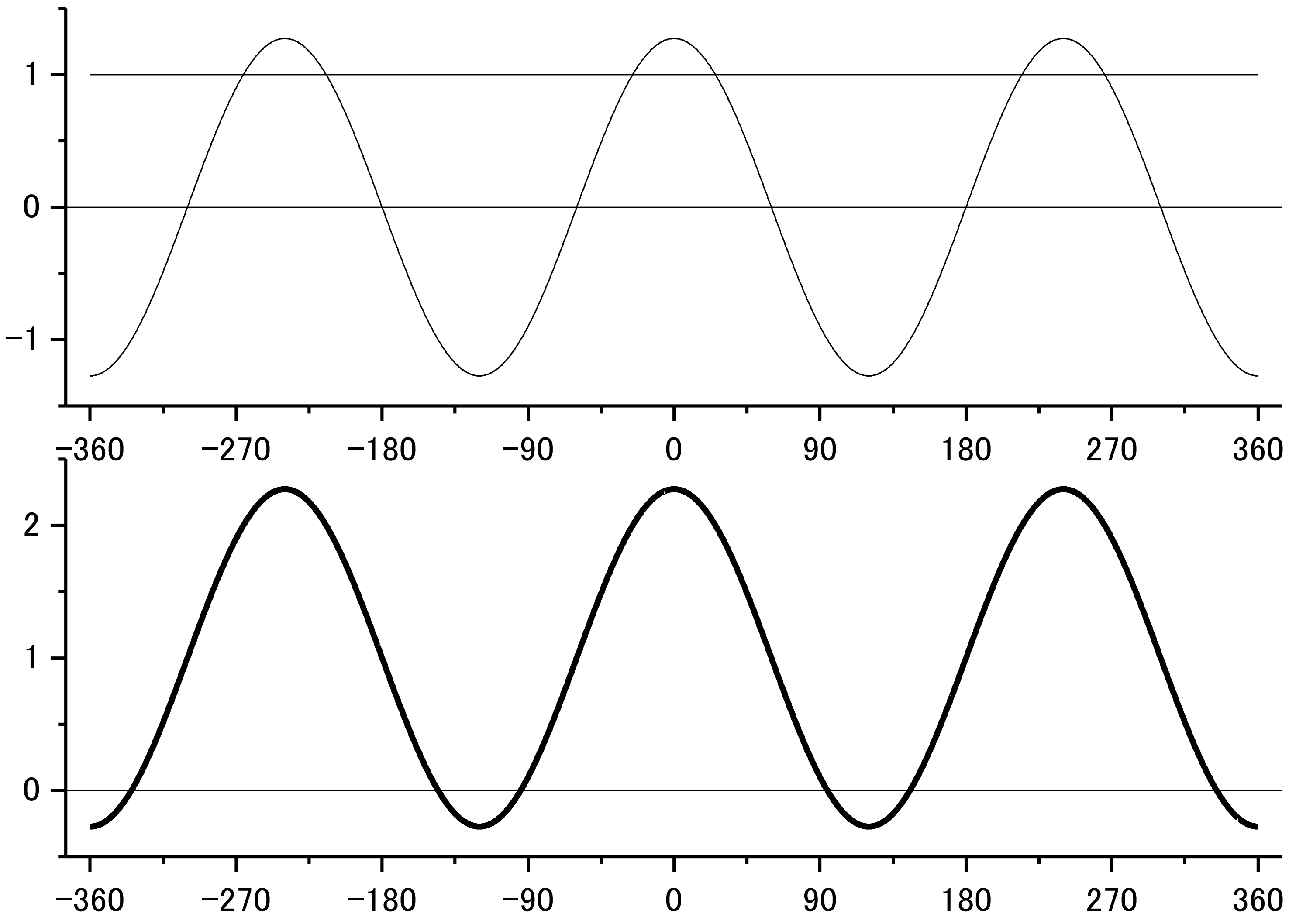

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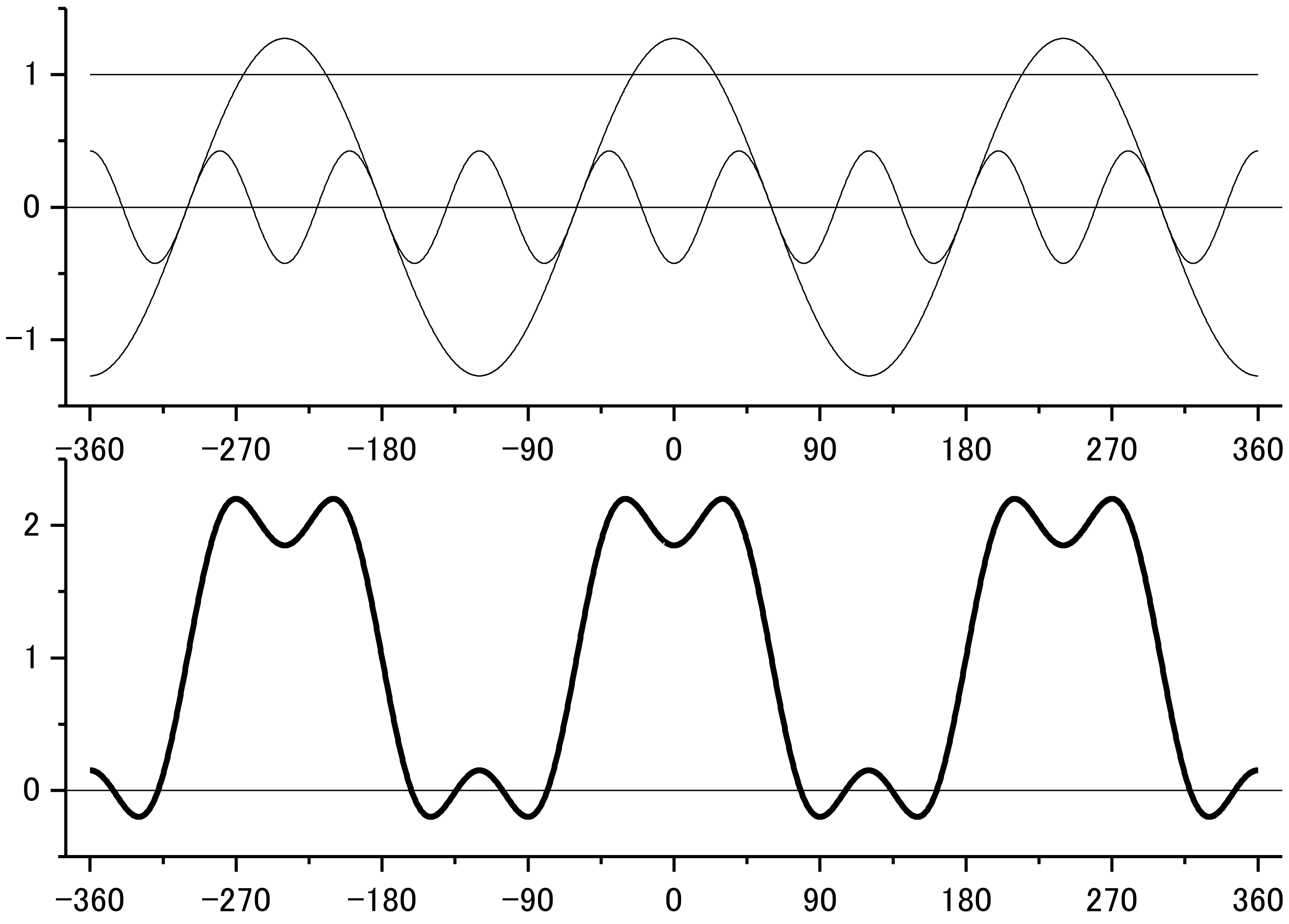
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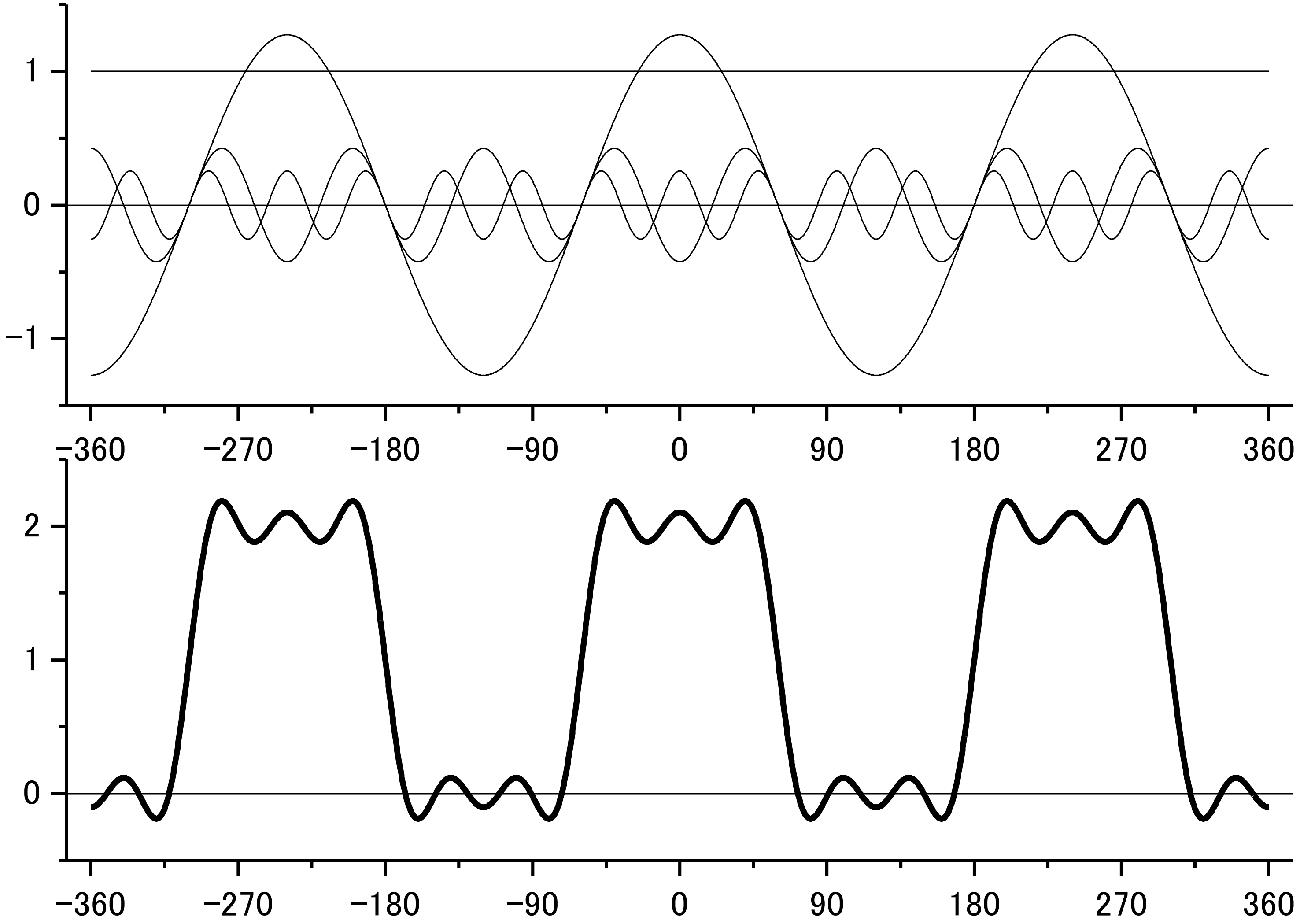

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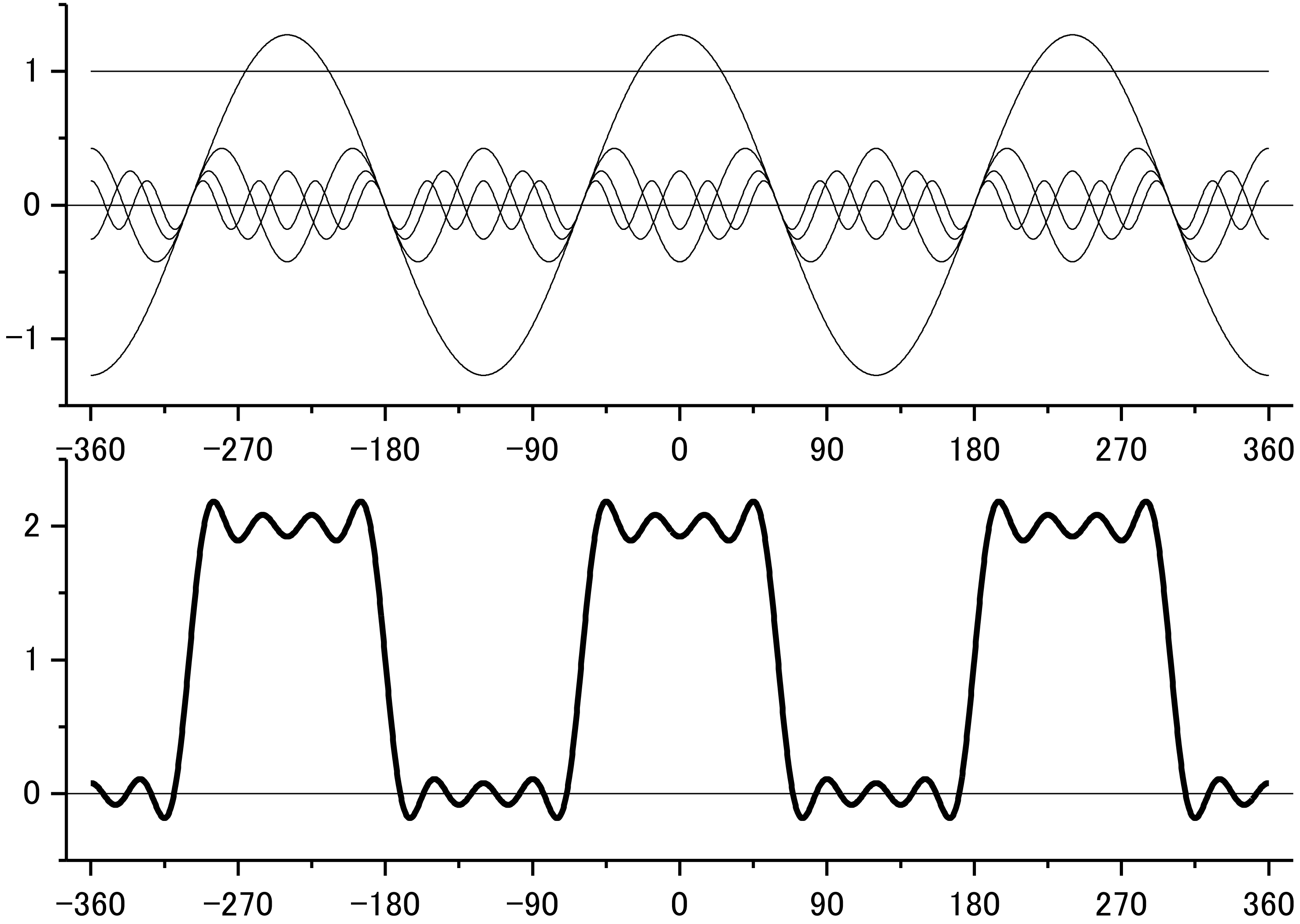
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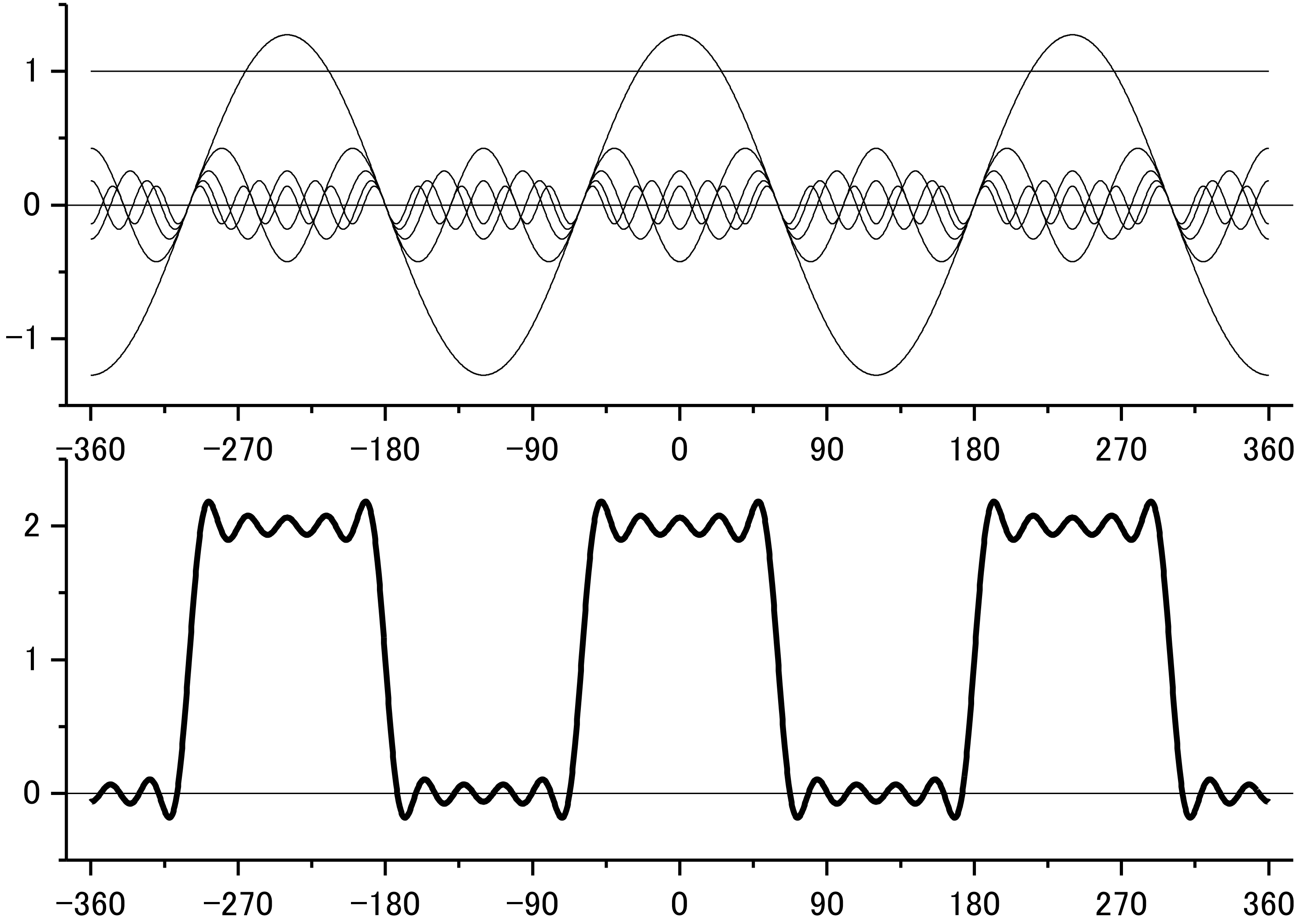
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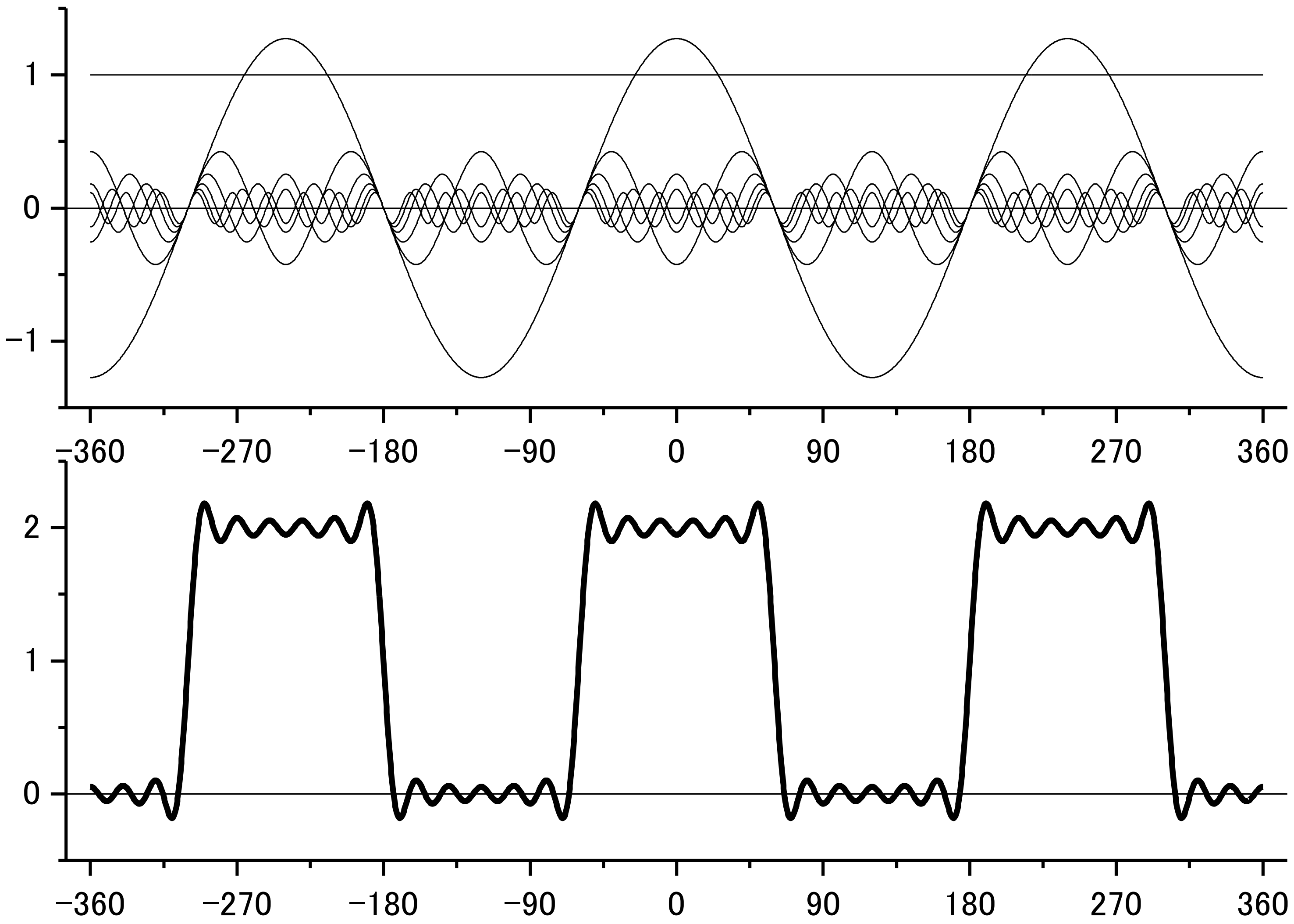
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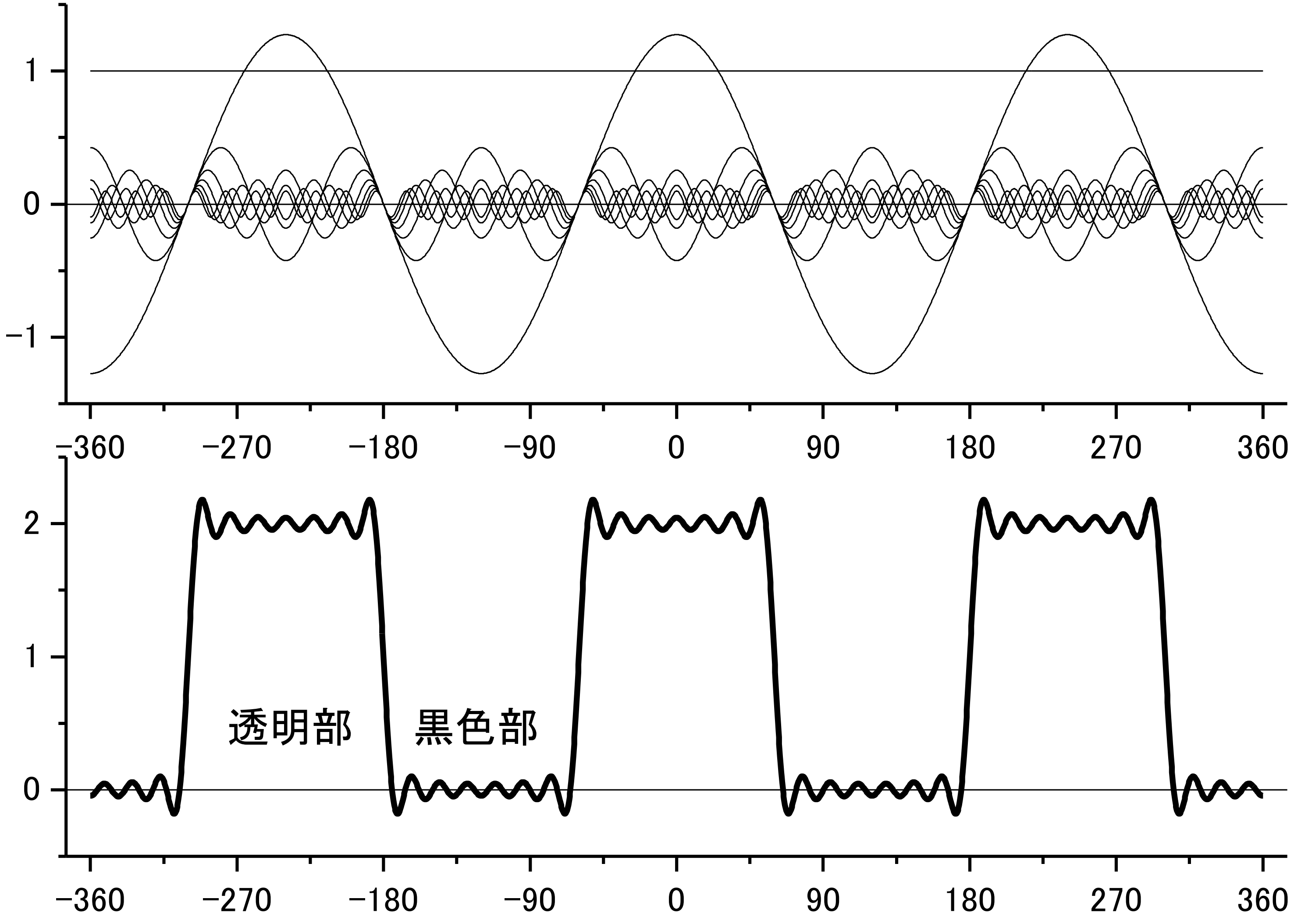
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回折格子3のパターン

$$f(\theta) = 1 + \frac{4}{\pi}\left(\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \frac{1}{7}\cos 7\theta + \cdots\right)$$

$$\theta = \frac{\pi}{d}x$$

$$f(\theta) = e^{i0x} + \frac{2}{\pi}\left[(e^{i\frac{\pi}{d}x} + e^{-i\frac{\pi}{d}x}) - \frac{1}{3}(e^{i\frac{3\pi}{d}x} + e^{-i\frac{3\pi}{d}x}) + \frac{1}{5}(e^{i\frac{5\pi}{d}x} + e^{-i\frac{5\pi}{d}x}) - \frac{1}{7}(e^{i\frac{7\pi}{d}x} + e^{-i\frac{7\pi}{d}x}) + \cdots\right]$$

$$+ \frac{1}{5}(e^{i\frac{5\pi}{d}x} + e^{-i\frac{5\pi}{d}x}) - \frac{1}{7}(e^{i\frac{7\pi}{d}x} + e^{-i\frac{7\pi}{d}x}) + \cdots\Big]$$

周期 $2d$

$$k_{2d} = \frac{2\pi}{2d} = \frac{\pi}{d}$$

$$K = 0, \pm k_{2d}, \pm 3k_{2d}, \pm 5k_{2d}, \pm 7k_{2d}, \cdots$$

フーリエ級数

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{d} + b_n \sin\frac{n\pi x}{d} \right)$$

周期$2d$

$$a_n = \frac{1}{d}\int_{-d}^{d} f(x)\cos\frac{n\pi x}{d}dx \quad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{d}\int_{-d}^{d} f(x)\sin\frac{n\pi x}{d}dx \quad n = 1, 2, \cdots$$

フーリエ変換

$$F(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

$$F(k)=\int_{-\infty}^{\infty} f(x)(\cos kx - i\sin kx)dx$$

$$=\int_{-\infty}^{\infty} f(x)\cos kx dx - i\int_{-\infty}^{\infty} f(x)\sin kx dx = A(k)-iB(k)$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[A(k)-iB(k)](\cos kx + i\sin kx)dk$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}[A(k)\cos kx + B(k)\sin kx]dk$$

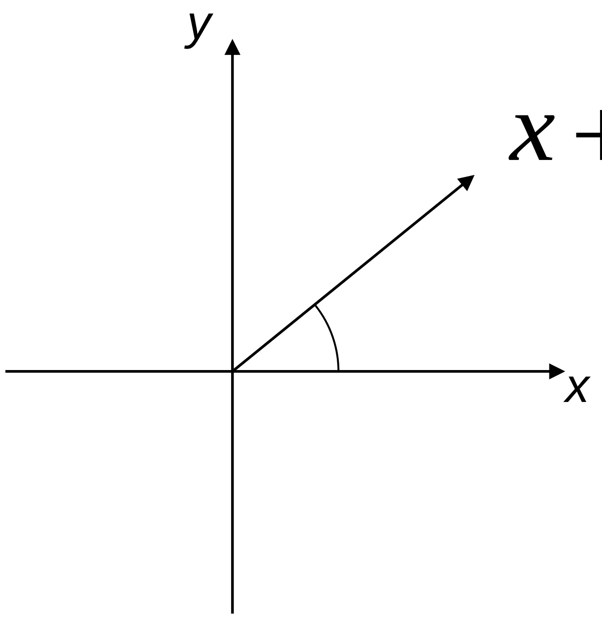
$$+i\frac{1}{2\pi}\int_{-\infty}^{\infty}[A(k)\sin kx - B(k)\cos kx]dk$$

$f(x)$が実関数ならば

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[A(k)\cos kx + B(k)\sin kx]dk$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{d}+b_n\sin\frac{n\pi x}{d}\right)$$ と比較せよ $k_n=\dfrac{n\pi}{d}$

$\cos\theta$と$\sin\theta$は直交(直交関数) 物理数学2



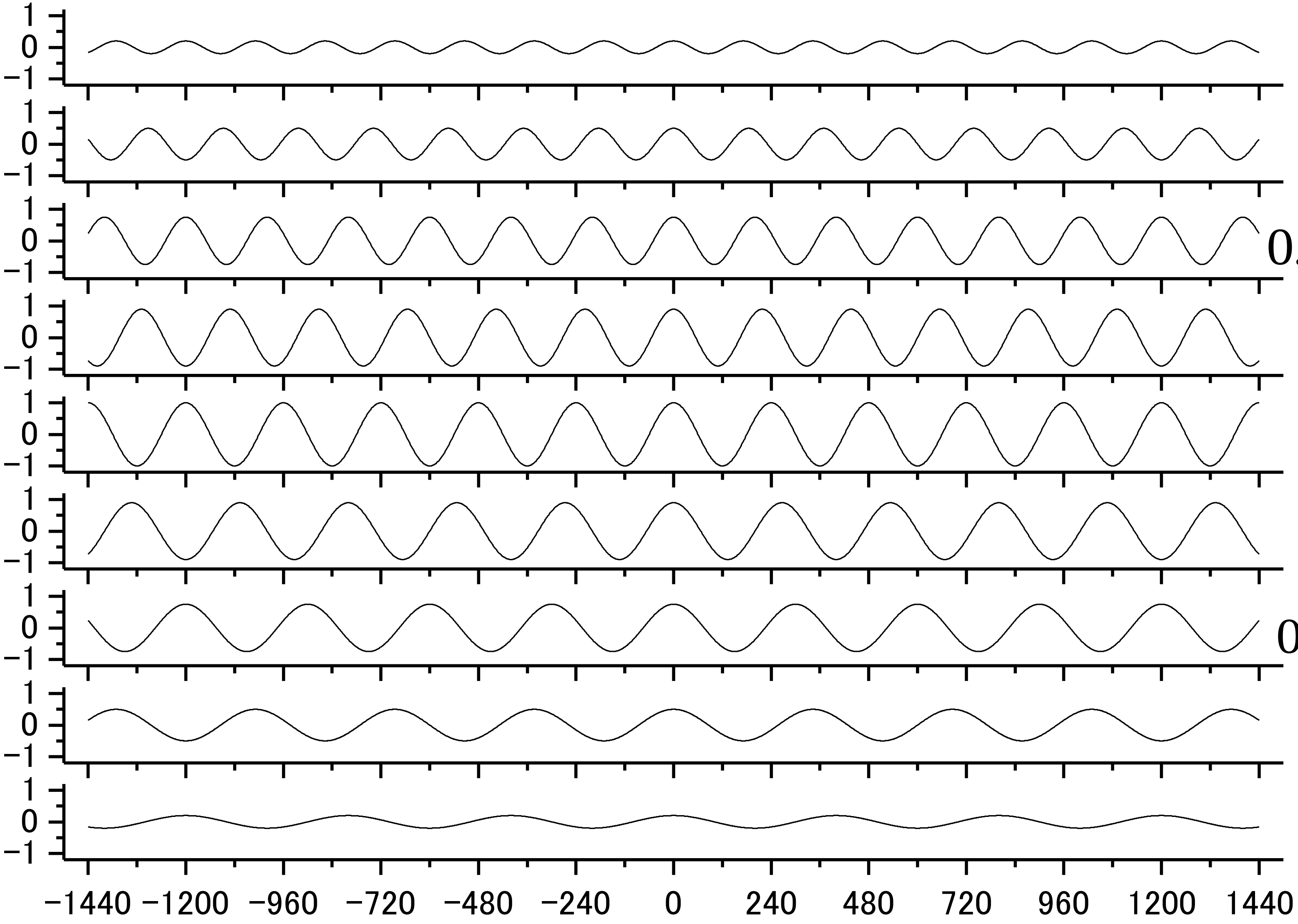
$$x + iy = \cos\theta + i\sin\theta = e^{i\theta}$$

$$R\cos(kx+\phi) = R(\cos kx\cos\phi - \sin kx\sin\phi)$$
$$= R(a\cos kx + b\sin kx)$$
$$= A\cos kx + B\sin kx$$

$\cos kx$に$\sin kx$を加えることは、任意の位相シフトϕを加えることに相当

こうすることで、フーリエ展開で$\cos kx$だけでは表せない関数も表せる

フーリエ級数でパルス波を作る



$0.2\cos(1.4\omega t)$

$0.5\cos(1.3\omega t)$

$0.75\cos(1.2\omega t)$

$0.9\cos(1.1\omega t)$

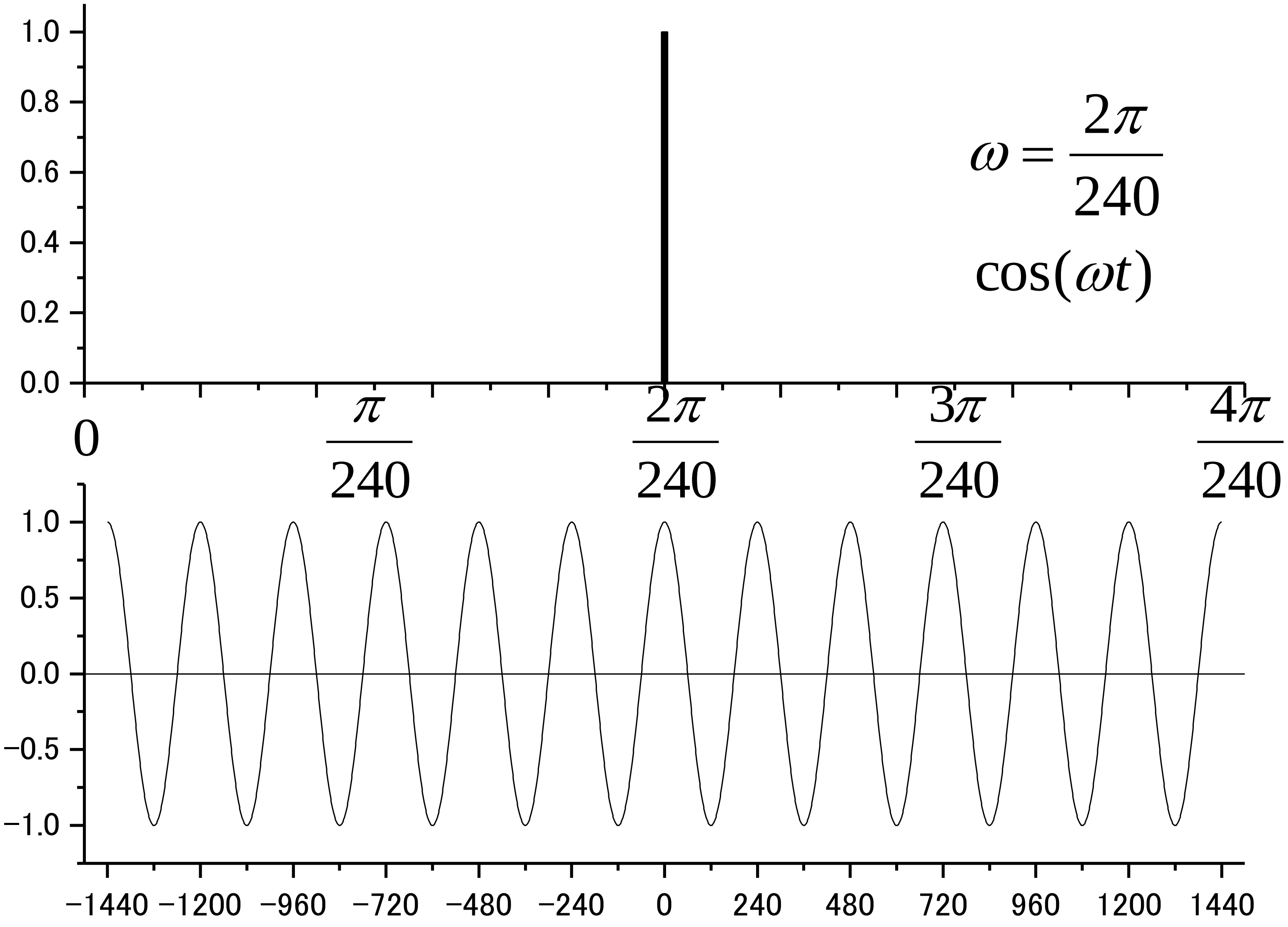
$\cos(\omega t)$

$0.9\cos(0.9\omega t)$

$0.75\cos(0.8\omega t)$

$0.5\cos(0.7\omega t)$

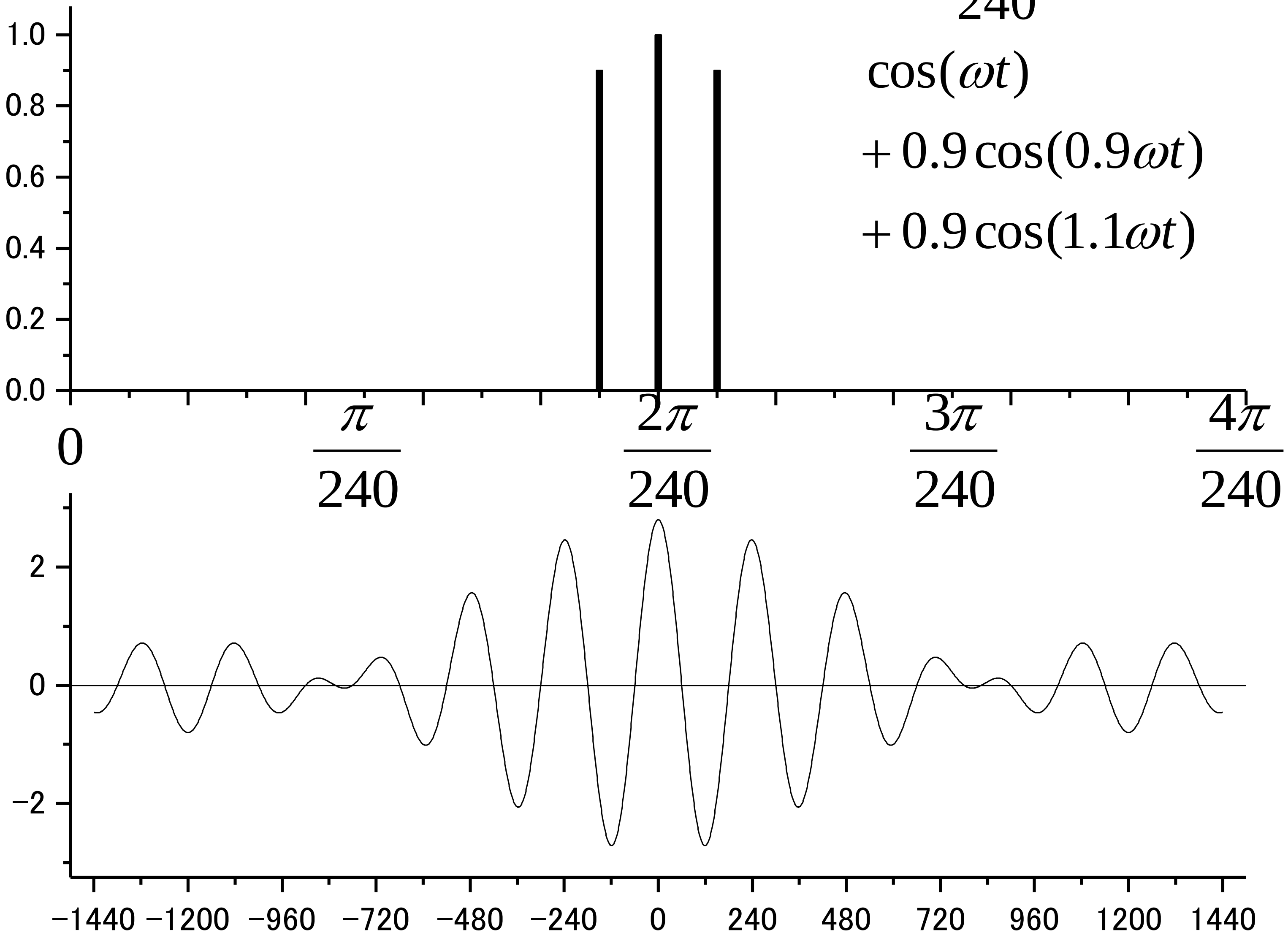
$0.2\cos(0.6\omega t)$

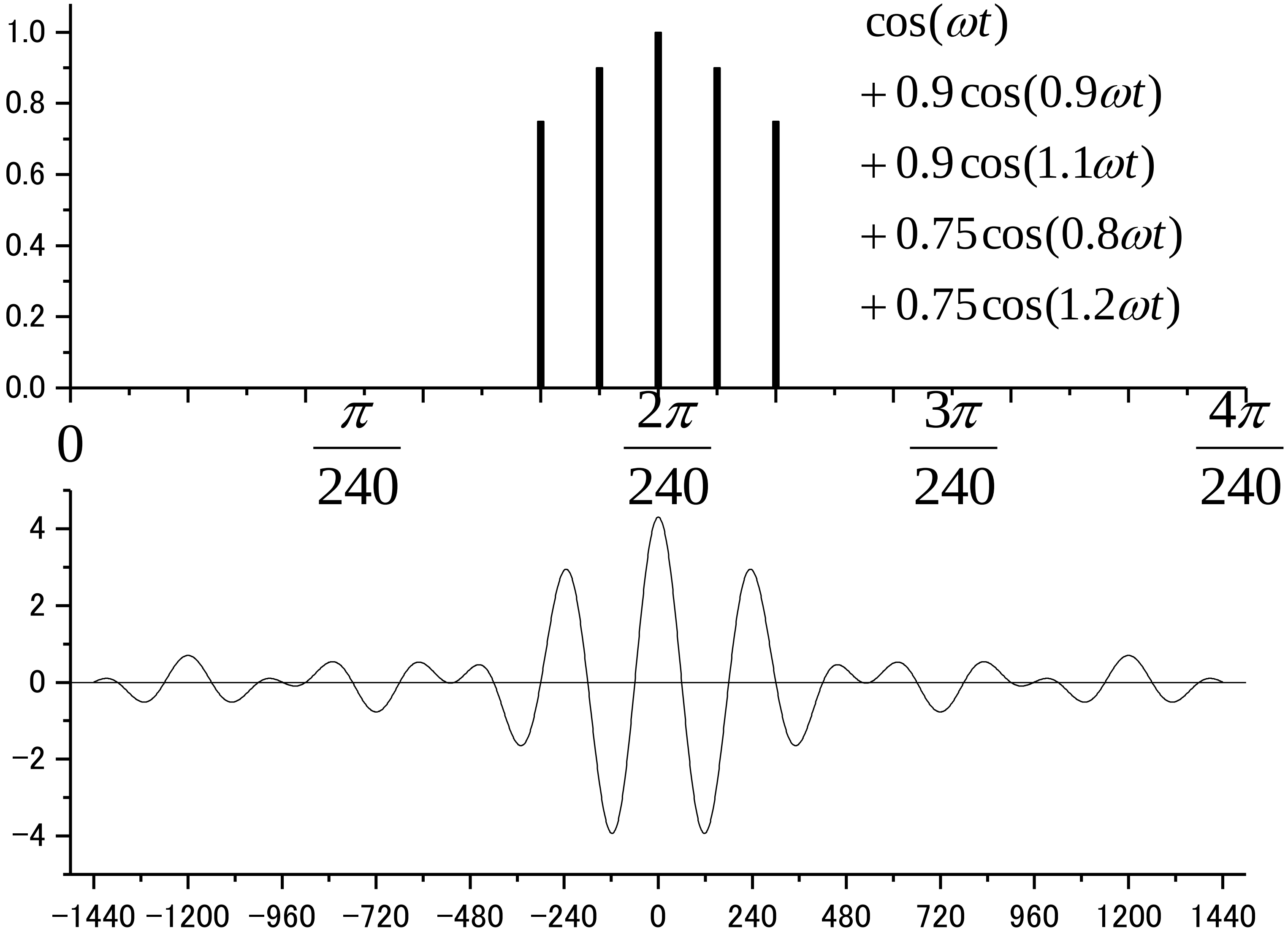


1.0
0.8
0.6
0.4
0.2
0.0
$\omega = \frac{2\pi}{240}$
$\cos(\omega t)$
0
$\frac{\pi}{240}$
$\frac{2\pi}{240}$
$\frac{3\pi}{240}$
$\frac{4\pi}{240}$
1.0
0.5
0.0
−0.5
−1.0
−1440 −1200 −960 −720 −480 −240 0 240 480 720 960 1200 1440

$$\omega = \frac{2\pi}{240}$$

$$\cos(\omega t) + 0.9\cos(0.9\omega t) + 0.9\cos(1.1\omega t)$$



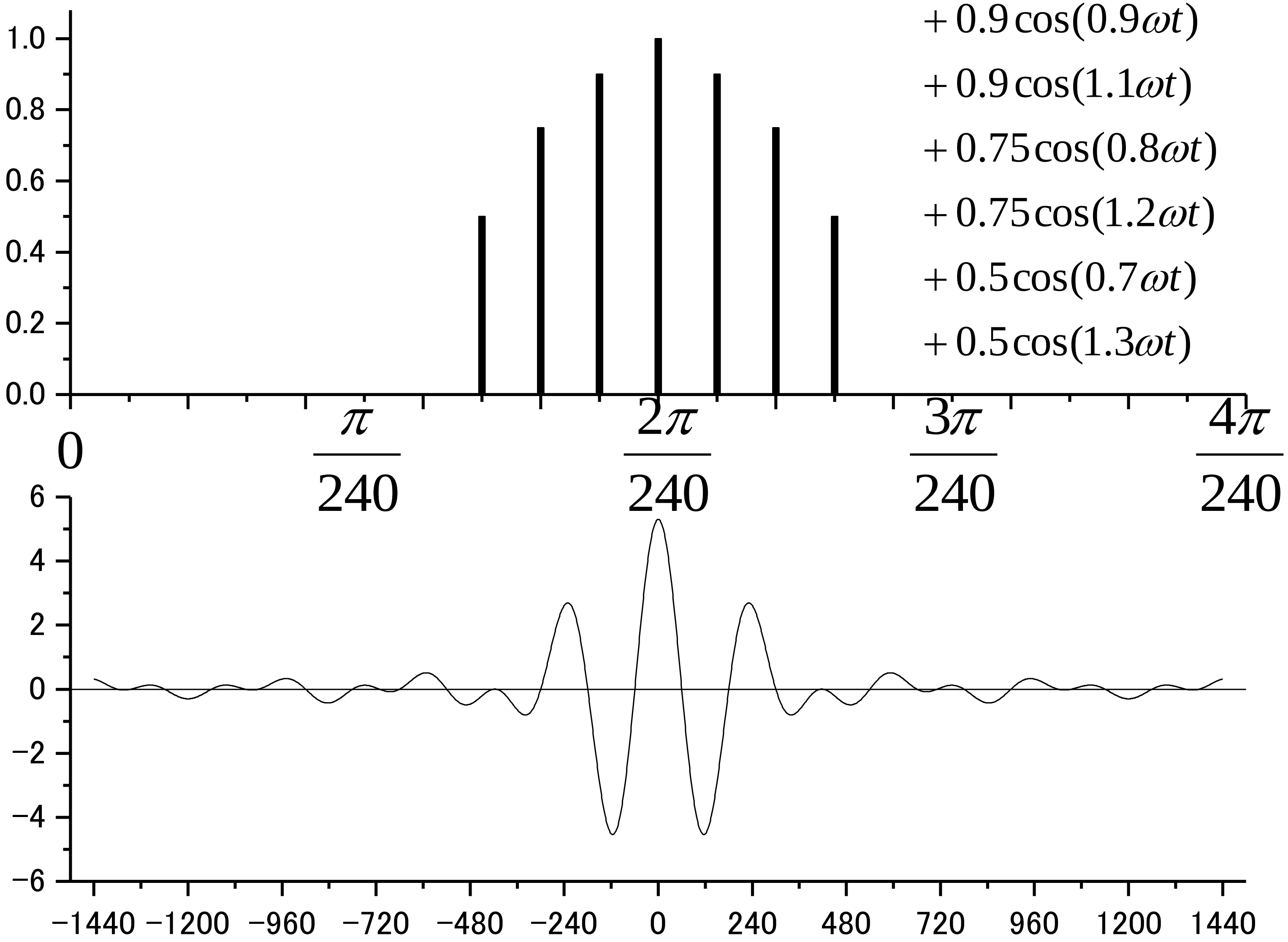


$$\omega = \frac{2\pi}{240}$$

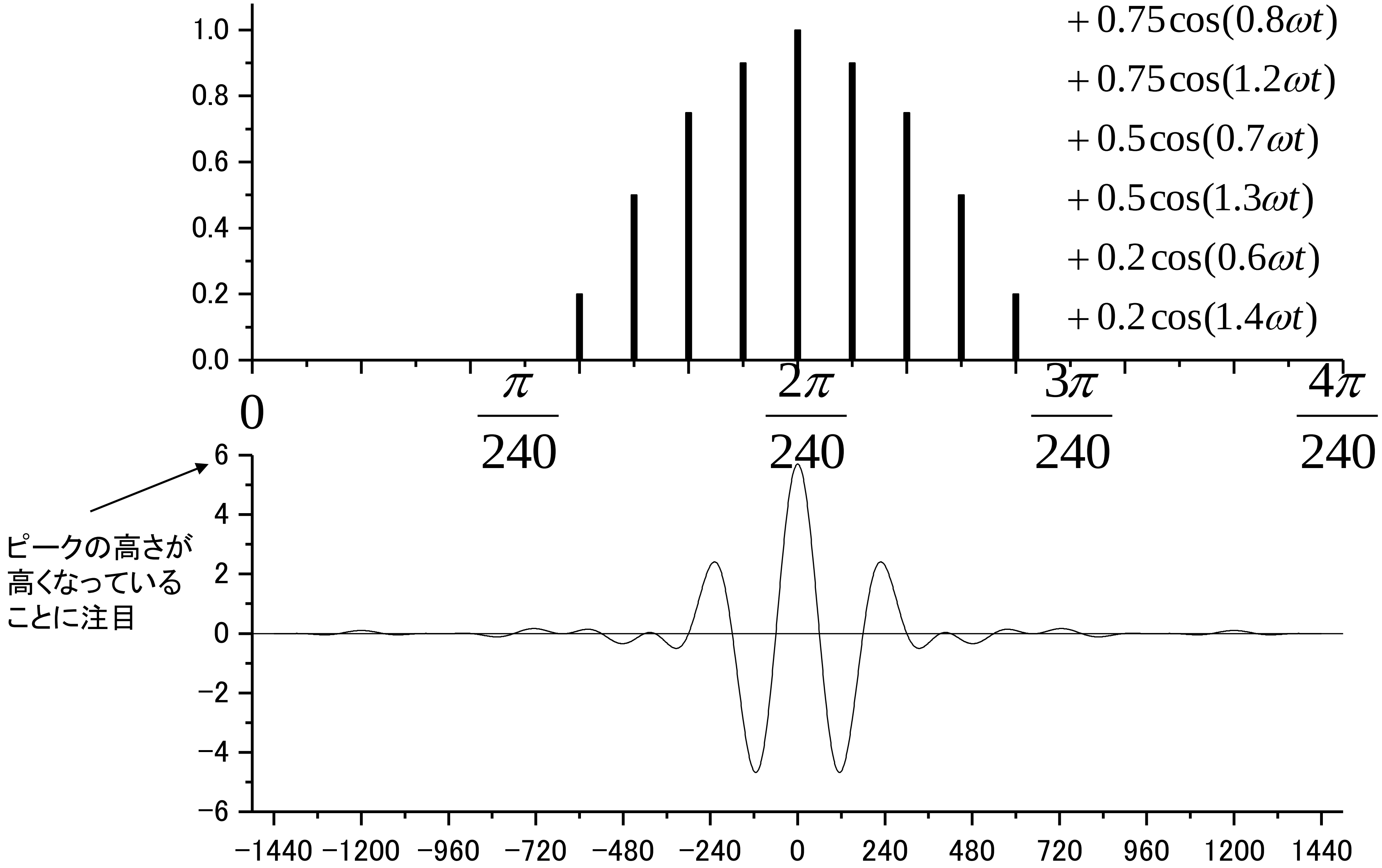
$$\cos(\omega t) + 0.9\cos(0.9\omega t) + 0.9\cos(1.1\omega t) + 0.75\cos(0.8\omega t) + 0.75\cos(1.2\omega t)$$

$$\omega = \frac{2\pi}{240}$$

$$\cos(\omega t)$$
$$+ 0.9\cos(0.9\omega t)$$
$$+ 0.9\cos(1.1\omega t)$$
$$+ 0.75\cos(0.8\omega t)$$
$$+ 0.75\cos(1.2\omega t)$$
$$+ 0.5\cos(0.7\omega t)$$
$$+ 0.5\cos(1.3\omega t)$$



$$\cos(\omega t) + 0.9\cos(0.9\omega t) + 0.9\cos(1.1\omega t) + 0.75\cos(0.8\omega t) + 0.75\cos(1.2\omega t) + 0.5\cos(0.7\omega t) + 0.5\cos(1.3\omega t) + 0.2\cos(0.6\omega t) + 0.2\cos(1.4\omega t)$$



Mode locking 位相同期法

一つのレーザー共振器からレーザー発振するさまざまな
波長成分の光の位相をそろえる技術

超短パルスの生成

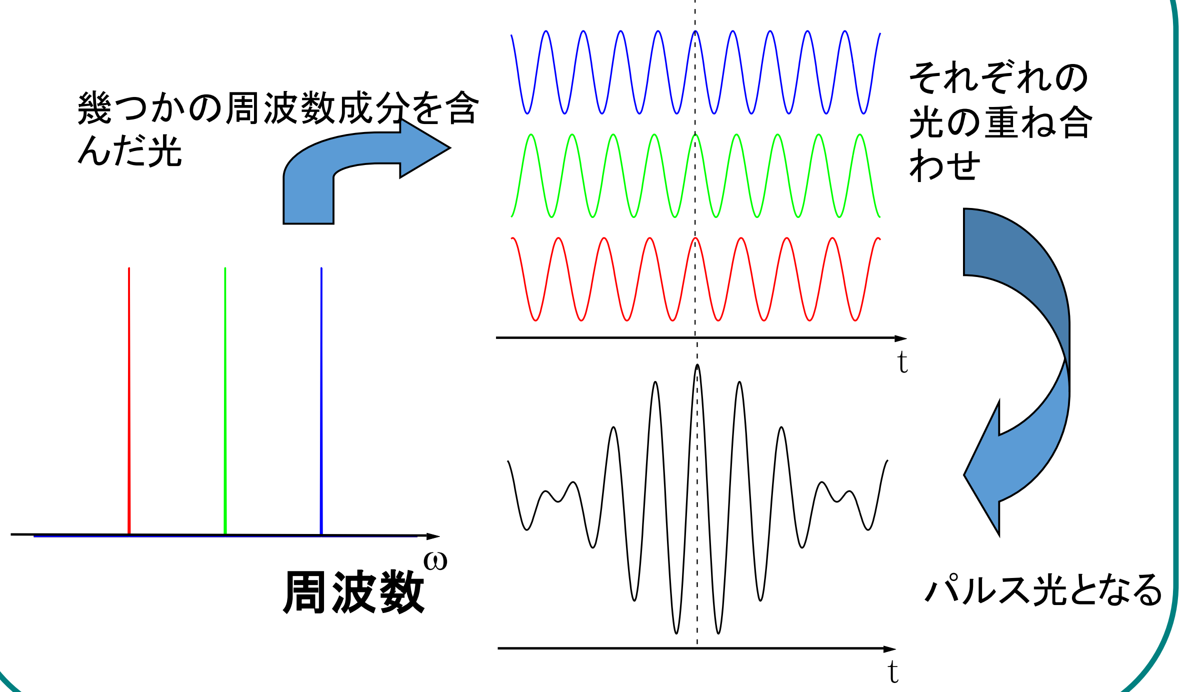
エネルギーが短い時間に集中　さまざまな利用価値がある

フェムト秒パルス

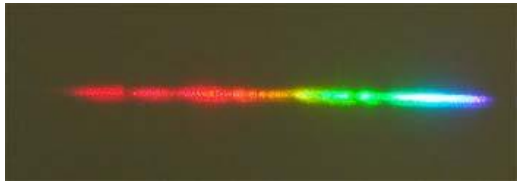
可視光パルスでは世界最短のサブ5fsの超短パルス

1フェムト秒（1fs）・・・10^{-15}s

パルスレーザー光とは？



広帯域の可視光を用いて、パルスを作る。

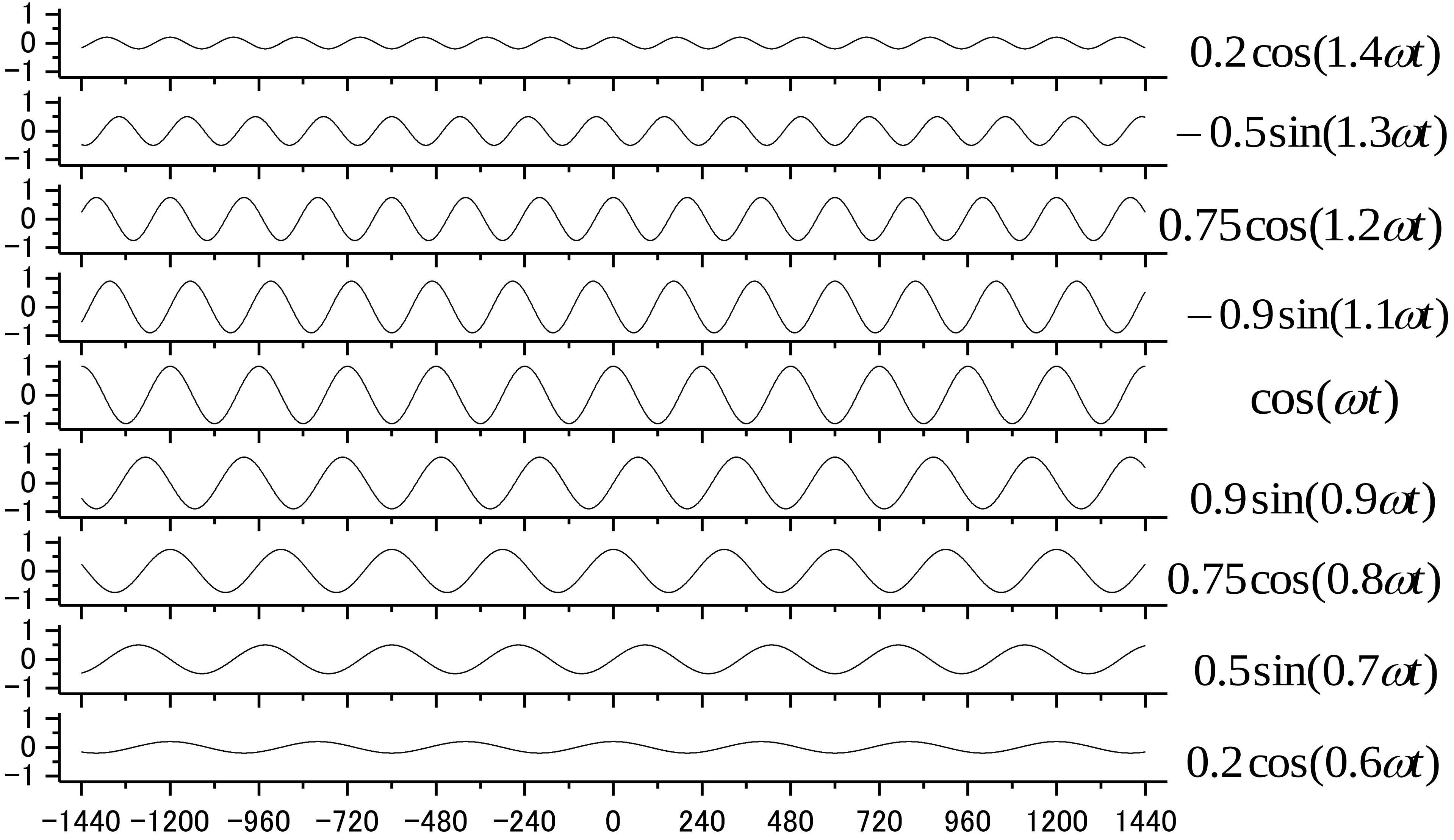


時間軸では、2,3周期の振動しかパルスの中に含まない。

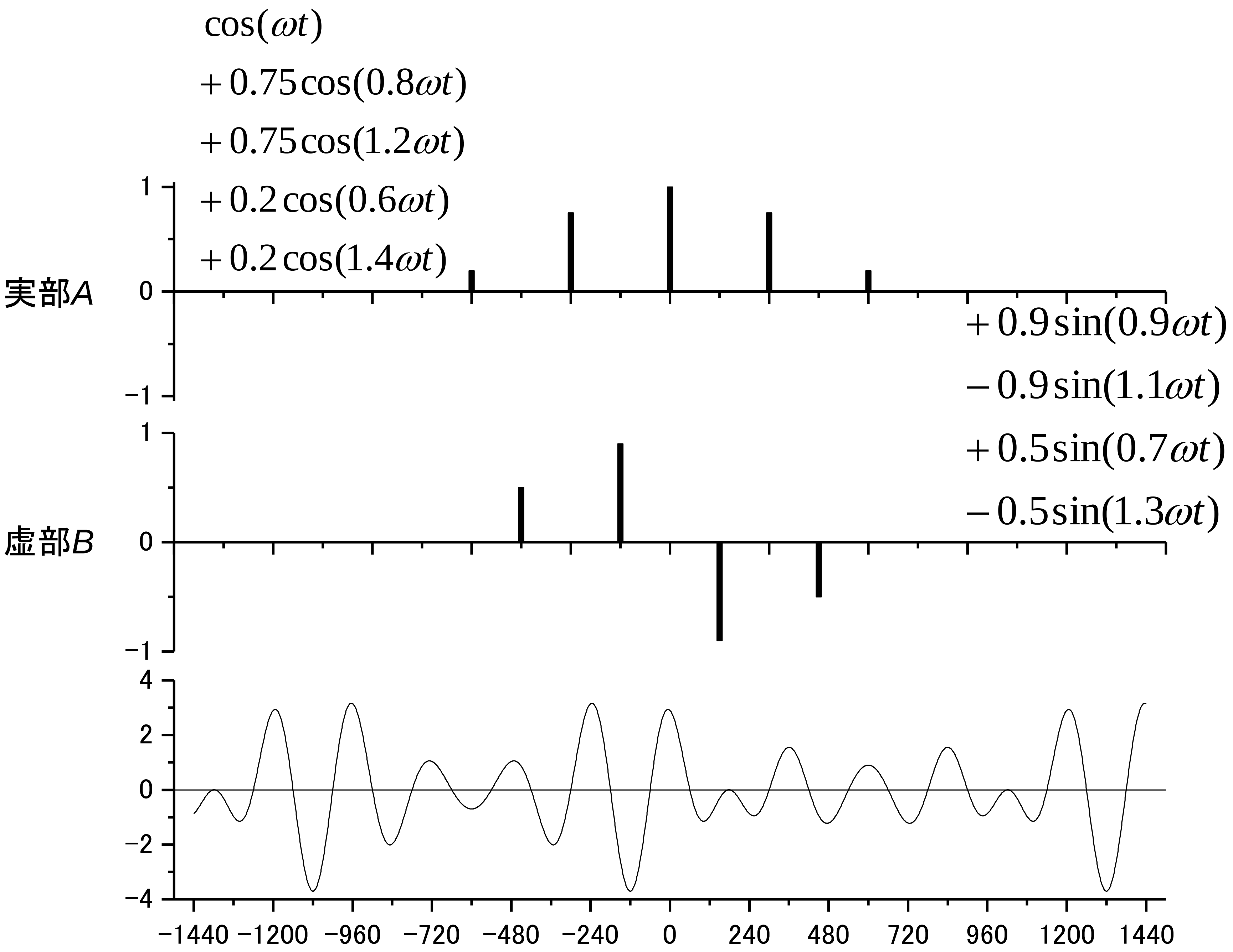
可視光パルスでは世界最短であるサブ5fsの超短パルスを実現！

東大（電通大）小林$_{孝}$研究室

位相がそろわないと？



1
0
−1
0.2cos(1.4ωt)
−0.5sin(1.3ωt)
0.75cos(1.2ωt)
−0.9sin(1.1ωt)
cos(ωt)
0.9sin(0.9ωt)
0.75cos(0.8ωt)
0.5sin(0.7ωt)
0.2cos(0.6ωt)
−1440 −1200 −960 −720 −480 −240 0 240 480 720 960 1200 1440


$\cos(\omega t)$
$+0.75\cos(0.8\omega t)$
$+0.75\cos(1.2\omega t)$
$+0.2\cos(0.6\omega t)$
$+0.2\cos(1.4\omega t)$
$+0.9\sin(0.9\omega t)$
$-0.9\sin(1.1\omega t)$
$+0.5\sin(0.7\omega t)$
$-0.5\sin(1.3\omega t)$
実部A
虚部B
1
0
−1
4
2
−2
−4
−1440
−1200
−960
−720
−480
−240
240
480
720
960
1200
1440

$$F(\omega) = A(\omega) + iB(\omega)$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} [A(\omega) + iB(\omega)][\cos\omega t + i\sin\omega t] d\omega$$

$$= \int_{-\infty}^{\infty} \{[A(\omega)\cos\omega t - B(\omega)\sin\omega t] + i[A(\omega)\sin\omega t + B(\omega)\cos\omega t]\} d\omega$$

$$= \int_{-\infty}^{\infty} [A(\omega)\cos\omega t - B(\omega)\sin\omega t] d\omega$$ 実関数

$$A(\omega) = \delta(\omega - \omega_0) + 0.75\delta(\omega - 0.8\omega_0) + 0.75\delta(\omega - 1.2\omega_0) + 0.2\delta(\omega - 0.6\omega_0) + 0.2\delta(\omega - 1,4\omega_0)$$

$$B(\omega) = -0.9\delta(\omega - 0.9\omega_0) + 0.9\delta(\omega - 1.1\omega_0) - 0.5\delta(\omega - 0.7\omega_0) + 0.5\delta(\omega - 1.3\omega_0)$$

実部A(ω)が偶関数（cos）,虚部B(ω)が奇関数（sin）