遅延ポテンシャルが波動方程式の解であることの証明　p.424

$$\boxed{V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}d\boldsymbol{r}'}$$

$$t_r=t-\frac{R}{c}$$

$$\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'=(x-x',y-y',z-z')$$

$$\nabla=\frac{\partial}{\partial\boldsymbol{r}}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$\frac{\partial}{\partial x}R=\frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}=\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{1}{2}}\cdot 2(x-x')$$

より $\dfrac{\partial}{\partial\boldsymbol{r}}R=\dfrac{\boldsymbol{R}}{R}=\hat{\boldsymbol{R}}$

$$\frac{\partial}{\partial\boldsymbol{r}}R^{-1}=\frac{\partial R}{\partial\boldsymbol{r}}\frac{\partial}{\partial R}R^{-1}=-R^{-2}\hat{\boldsymbol{R}}$$

$$\boxed{\nabla V(\boldsymbol{r},t)=\frac{\partial}{\partial\boldsymbol{r}}V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left[(\frac{\partial}{\partial\boldsymbol{r}}\rho(\boldsymbol{r}',t_r))\frac{1}{R}+\rho\frac{\partial}{\partial\boldsymbol{r}}\left(\frac{1}{R}\right)\right]d\boldsymbol{r}'=\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\frac{\hat{\boldsymbol{R}}}{R}\frac{\partial\rho}{\partial t}-\rho\frac{\hat{\boldsymbol{R}}}{R^2}\right]d\boldsymbol{r}'}$$

$$\nabla\rho=\frac{\partial}{\partial\boldsymbol{r}}\rho(\boldsymbol{r}',t_r)=\frac{\partial t_r}{\partial\boldsymbol{r}}\frac{\partial}{\partial t_r}\rho=-\frac{1}{c}\hat{\boldsymbol{R}}\frac{\partial}{\partial t}\rho$$

$$\because\frac{\partial t_r}{\partial\boldsymbol{r}}=\frac{\partial}{\partial\boldsymbol{r}}\left(t-\frac{R}{c}\right)=-\frac{1}{c}\frac{\partial}{\partial\boldsymbol{r}}R=-\frac{1}{c}\hat{\boldsymbol{R}}$$

$$\frac{\partial}{\partial t}\rho=\frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r}\rho=\frac{\partial}{\partial t_r}\rho\quad\because\frac{\partial t_r}{\partial t}=1$$

$$\boxed{\begin{aligned}\nabla^2V(\boldsymbol{r},t)&=\frac{1}{4\pi\varepsilon_0}\int\nabla\cdot\left[-\frac{1}{c}\frac{\hat{\boldsymbol{R}}}{R}\frac{\partial\rho}{\partial t}-\rho\frac{\hat{\boldsymbol{R}}}{R^2}\right]d\boldsymbol{r}'=\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R}\frac{\partial\rho}{\partial t}\right)-\nabla\cdot\left(\rho\frac{\hat{\boldsymbol{R}}}{R^2}\right)\right]d\boldsymbol{r}'\\&=\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R}\right)\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\hat{\boldsymbol{R}}}{R}\cdot\nabla\left(\frac{\partial\rho}{\partial t}\right)-(\nabla\rho)\cdot\frac{\hat{\boldsymbol{R}}}{R^2}-\rho\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R^2}\right)\right]d\boldsymbol{r}'\end{aligned}}$$

$$\nabla\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial\boldsymbol{r}}\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t}=\frac{\partial t_r}{\partial\boldsymbol{r}}\frac{\partial}{\partial t_r}\frac{\partial\rho}{\partial t}=-\frac{1}{c}\hat{\boldsymbol{R}}\frac{\partial^2\rho}{\partial t^2}$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R}\right)=\hat{\boldsymbol{R}}\cdot\nabla\left(\frac{1}{R}\right)+\frac{1}{R}\nabla\cdot\left(\hat{\boldsymbol{R}}\right)=\hat{\boldsymbol{R}}\cdot\left(-\frac{\hat{\boldsymbol{R}}}{R^2}\right)+\frac{1}{R}\nabla\cdot\left(\frac{\boldsymbol{R}}{R}\right)=-\frac{1}{R^2}+\frac{1}{R}\left[\frac{1}{R}\nabla\cdot\boldsymbol{R}+\boldsymbol{R}\cdot\nabla\left(\frac{1}{R}\right)\right]$$

$$=-\frac{1}{R^2}+\frac{1}{R}\left[\frac{3}{R}-\boldsymbol{R}\cdot\frac{\hat{\boldsymbol{R}}}{R^2}\right]=-\frac{1}{R^2}+\frac{1}{R}\left[\frac{3}{R}-\frac{1}{R}\right]=\frac{1}{R^2}$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R^2}\right)=4\pi\delta^3(\boldsymbol{R})$$

$\because$ **ガウスの積分** $\displaystyle\iint_S\frac{\hat{\boldsymbol{R}}}{R^2}\cdot\boldsymbol{n}dS=\iint_S\frac{\hat{\boldsymbol{R}}}{R^2}\cdot\hat{\boldsymbol{R}}R^2d\Omega=\iint_S d\Omega=4\pi\quad\iiint_V\nabla\cdot\left(\frac{\hat{\boldsymbol{R}}}{R^2}\right)d\boldsymbol{R}=\iint_S\frac{\hat{\boldsymbol{R}}}{R^2}\cdot\boldsymbol{n}dS$ **より**

$$\boxed{\begin{aligned}\nabla^2V(\boldsymbol{r},t)&=\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\left(\frac{1}{R^2}\right)\frac{\partial\rho}{\partial t}-\frac{1}{c}\frac{\hat{\boldsymbol{R}}}{R}\cdot\left(-\frac{1}{c}\hat{\boldsymbol{R}}\frac{\partial^2\rho}{\partial t^2}\right)-(-\frac{1}{c}\hat{\boldsymbol{R}}\frac{\partial}{\partial t}\rho)\cdot\frac{\hat{\boldsymbol{R}}}{R^2}-\rho4\pi\delta^3(\boldsymbol{R})\right]d\boldsymbol{r}'\\&=\frac{1}{4\pi\varepsilon_0}\int\left[\frac{1}{c^2}\frac{1}{R}\frac{\partial^2\rho}{\partial t^2}-4\pi\rho\delta^3(\boldsymbol{r}-\boldsymbol{r}')\right]d\boldsymbol{r}'=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{R}d\boldsymbol{r}'-\frac{1}{\varepsilon_0}\rho(\boldsymbol{r},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}V-\frac{1}{\varepsilon_0}\rho\end{aligned}}$$

遅延ポテンシャルがローレンツゲージ条件を満たすことの証明　Prob.10.8

$V, \boldsymbol{A}$を

$$V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}d\boldsymbol{r}'=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}\delta(t_r-t+\frac{R}{c})dt_r d\boldsymbol{r}'=\frac{1}{4\pi\varepsilon_0}\int\left[\int_{-\infty}^{\infty}\frac{\rho(\boldsymbol{r}',t_r)}{R}\delta(t_r-t+\frac{R}{c})dt_r\right]d\boldsymbol{r}'$$

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{R}d\boldsymbol{r}'=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{R}\delta(t_r-t+\frac{R}{c})dt_r d\boldsymbol{r}'$$

と書き直す。すると、t_rは積分変数となり、$\frac{\partial}{\partial t},\frac{\partial}{\partial \boldsymbol{r}}=\nabla$が直接作用しない

$$t_r=t-\frac{R}{c}\quad \boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'=(x-x',y-y',z-z')\quad \nabla=\frac{\partial}{\partial\boldsymbol{r}}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\quad R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$\frac{\partial V}{\partial t}=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}\frac{\partial\delta(t_r-t+\frac{R}{c})}{\partial t}dt_r d\boldsymbol{r}'$$

$$=-\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}\frac{\partial\delta(t_r-t+\frac{R}{c})}{\partial t_r}dt_r d\boldsymbol{r}'$$

$$=-\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t_r)}{R}\left[\delta(t_r-t+\frac{R}{c})\right]_{-\infty}^{\infty}d\boldsymbol{r}'+\frac{1}{4\pi\varepsilon_0}\int\frac{1}{R}\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t_r}\delta(t_r-t+\frac{R}{c})dt_r d\boldsymbol{r}'\quad \text{部分積分}$$

$$=\frac{1}{4\pi\varepsilon_0}\int\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t_r}\frac{\delta(t_r-t+\frac{R}{c})}{R}dt_r d\boldsymbol{r}'$$

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\int\boldsymbol{J}(\boldsymbol{r}',t_r)\cdot\nabla\left(\frac{\delta(t_r-t+\frac{R}{c})}{R}\right)dt_r d\boldsymbol{r}'=-\frac{\mu_0}{4\pi}\int\boldsymbol{J}(\boldsymbol{r}',t_r)\cdot\nabla_{\boldsymbol{r}'}\left(\frac{\delta(t_r-t+\frac{R}{c})}{R}\right)dt_r d\boldsymbol{r}'$$

$$=\frac{\mu_0}{4\pi}\int[\nabla_{\boldsymbol{r}'}\cdot\boldsymbol{J}(\boldsymbol{r}',t_r)]\frac{\delta(t_r-t+\frac{R}{c})}{R}dt_r d\boldsymbol{r}'\quad \text{部分積分}$$

ローレンツ条件$\nabla\cdot\boldsymbol{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}=\frac{\mu_0}{4\pi}\int[\nabla_{\boldsymbol{r}'}\cdot\boldsymbol{J}(\boldsymbol{r}',t_r)+\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t_r}]\frac{\delta(t_r-t+\frac{R}{c})}{R}dt_r d\boldsymbol{r}'=0$

$\because \nabla_{\boldsymbol{r}'}\cdot\boldsymbol{J}(\boldsymbol{r}',t_r)+\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t_r}=0$　電荷の連続方程式（電荷保存則）

$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)$の計算で

$$\nabla\cdot(f\boldsymbol{w})=(\nabla f)\cdot\boldsymbol{w}+f(\nabla\cdot\boldsymbol{w})$$

$$\iint_S(f\boldsymbol{w})\cdot\boldsymbol{n}dS=\iiint_V[\nabla\cdot(f\boldsymbol{w})]d\boldsymbol{r}=\iiint_V[(\nabla f)\cdot\boldsymbol{w}]d\boldsymbol{r}+\iiint_V[f(\nabla\cdot\boldsymbol{w})]d\boldsymbol{r}$$

$$\iiint_V[(\nabla f)\cdot\boldsymbol{w}]d\boldsymbol{r}=\iint_S(f\boldsymbol{w})\cdot\boldsymbol{n}dS-\iiint_V[f(\nabla\cdot\boldsymbol{w})]d\boldsymbol{r}$$

で、$\boldsymbol{w}=\boldsymbol{J},\ f=\frac{\delta}{R}, S$として無限遠

を使った。

$$\nabla(fg)=(\nabla f)g+f(\nabla g)$$

$$\iiint_V\nabla(fg)d\boldsymbol{r}=?$$

運動する点電荷の遅延ポテンシャル（リエナール・ヴィーヒェルトポテンシャル）pp.429-432

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = c(t - t_r)$$

$$t_r = t - \frac{R}{c}$$

$\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{w}(t)) \quad \boldsymbol{w}(t)$: 点電荷の運動の軌跡

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}', t_r)}{R} d\boldsymbol{r}' = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}', t')}{R}\delta(t' - t + \frac{R}{c}) dt' d\boldsymbol{r}'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta(\boldsymbol{r}' - \boldsymbol{w}(t'))}{|\boldsymbol{r} - \boldsymbol{r}'|}\delta(t' - t + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}) dt' d\boldsymbol{r}'$$

$$= \frac{q}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r} - \boldsymbol{w}(t')|}\delta(t' - t + \frac{|\boldsymbol{r} - \boldsymbol{w}(t')|}{c}) dt'$$

$$f(t') \equiv t' - t + \frac{|\boldsymbol{r} - \boldsymbol{w}(t')|}{c}$$

$$\frac{df(t')}{dt'} = 1 + \frac{1}{c}\frac{\partial}{\partial t'}|\boldsymbol{r} - \boldsymbol{w}(t')| = 1 + \frac{1}{c}\frac{\partial}{\partial t'}\left[(\boldsymbol{r} - \boldsymbol{w}(t'))^2\right]^{1/2} = 1 + \frac{1}{c}\frac{d\boldsymbol{w}}{dt'}\frac{\partial}{\partial \boldsymbol{w}}\left[(\boldsymbol{r} - \boldsymbol{w}(t'))^2\right]^{1/2}$$

$$= 1 + \frac{1}{c}\frac{d\boldsymbol{w}}{dt'}\frac{\partial(\boldsymbol{r} - \boldsymbol{w})^2}{\partial \boldsymbol{w}}\frac{d\left[(\boldsymbol{r} - \boldsymbol{w})^2\right]^{1/2}}{d(\boldsymbol{r} - \boldsymbol{w})^2} = 1 + \frac{1}{c}\frac{d\boldsymbol{w}}{dt'}[-2(\boldsymbol{r} - \boldsymbol{w})]\frac{1}{2}\left[(\boldsymbol{r} - \boldsymbol{w})^2\right]^{-1/2}$$

$$= 1 - \frac{1}{c}\frac{(\boldsymbol{r} - \boldsymbol{w})}{|\boldsymbol{r} - \boldsymbol{w}|}\cdot\frac{d\boldsymbol{w}}{dt'} = 1 - \frac{1}{c}\frac{\boldsymbol{R}}{|\boldsymbol{R}|}\cdot\boldsymbol{v}(t') = 1 - \frac{1}{c}\hat{\boldsymbol{R}}\cdot\boldsymbol{v}(t')$$

$t' - t + \dfrac{|\boldsymbol{r} - \boldsymbol{w}(t')|}{c} = 0$の解は$t' = t_r$　このとき$\delta(f(t')) = \dfrac{\delta(t' - t_r)}{\left|\left(\dfrac{df(t')}{dt'}\right)_{t'=t_r}\right|}$より

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r} - \boldsymbol{w}(t')|}\frac{\delta(t' - t_r)}{\left|\left(\dfrac{df(t')}{dt'}\right)_{t'=t_r}\right|} dt' = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{r} - \boldsymbol{w}(t_r)|}\frac{1}{1 - \dfrac{1}{c}\dfrac{(\boldsymbol{r} - \boldsymbol{w}(t_r))}{|\boldsymbol{r} - \boldsymbol{w}(t_r)|}\cdot\left(\dfrac{d\boldsymbol{w}(t')}{dt'}\right)_{t'=t_r}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{r} - \boldsymbol{w}(t_r)| - \dfrac{1}{c}(\boldsymbol{r} - \boldsymbol{w}(t_r))\cdot\boldsymbol{v}(t_r)} = \frac{1}{4\pi\varepsilon_0}\frac{qc}{Rc - \boldsymbol{R}\cdot\boldsymbol{v}}$$