遅延ポテンシャルが波動方程式の解であることの証明 2　p.424 の別解

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V(\boldsymbol{r},t)=-\frac{1}{\varepsilon_0}\rho(\boldsymbol{r},t)$$ 　の解　を求める

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r},t)=-\delta(\boldsymbol{r})\delta(t)$$ 　の解 $G(\boldsymbol{r},t)$ が求まれば

$$V(\boldsymbol{r},t)=\frac{1}{\varepsilon_0}\int G(\boldsymbol{r}-\boldsymbol{r}',t-t')\rho(\boldsymbol{r}',t')d\boldsymbol{r}'dt'$$ 　と表せる。なぜなら

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V(\boldsymbol{r},t)=\frac{1}{\varepsilon_0}\int\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r}-\boldsymbol{r}',t-t')\rho(\boldsymbol{r}',t')d\boldsymbol{r}'dt'$$

$$=-\frac{1}{\varepsilon_0}\int\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')\rho(\boldsymbol{r}',t')d\boldsymbol{r}'dt'=-\frac{1}{\varepsilon_0}\rho(\boldsymbol{r},t)$$ 　だから。

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[i\omega t]d\omega\quad\delta(\boldsymbol{r})=\frac{1}{(2\pi)^3}\int\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]d\boldsymbol{k}$$

$$G(\boldsymbol{r},t)=\frac{1}{(2\pi)^4}\int G(\boldsymbol{k},\omega)\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r},t)=-\delta(\boldsymbol{r})\delta(t)$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int G(\boldsymbol{k},\omega)\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}=-\int\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}$$

$$\int G(\boldsymbol{k},\omega)\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}=-\int\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}$$

$$\int G(\boldsymbol{k},\omega)\left(-k^2+\frac{\omega^2}{c^2}\right)\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}=-\int\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}$$

$$G(\boldsymbol{k},\omega)=\frac{1}{k^2-\dfrac{\omega^2}{c^2}}$$

$$G(\boldsymbol{r},t)=\frac{1}{(2\pi)^4}\int G(\boldsymbol{k},\omega)\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})]d\omega d\boldsymbol{k}=\frac{1}{(2\pi)^4}\int\frac{1}{k^2-\dfrac{\omega^2}{c^2}}\exp[i(\omega t-kr\cos\theta)]k^2\sin\theta dk d\theta d\phi d\omega$$

$\cos\theta=u\quad-\sin\theta d\theta=du$

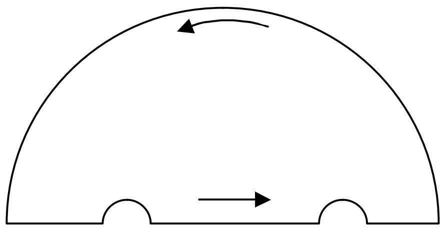
$$=-\frac{1}{(2\pi)^4}\int\frac{k^2}{k^2-\dfrac{\omega^2}{c^2}}\exp[i(\omega t-kru)]dkdud\phi d\omega$$

$$=-\frac{2\pi}{(2\pi)^4}\int_{-\infty}^{\infty}d\omega\exp[i\omega t]\int_0^{\infty}dk\frac{k^2}{k^2-\dfrac{\omega^2}{c^2}}\int_1^{-1}du\exp[ikru]=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d\omega\exp[i\omega t]\int_0^{\infty}dk\frac{k^2}{k^2-\dfrac{\omega^2}{c^2}}\frac{\exp[ikr]-\exp[-ikr]}{ikr}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \exp[i\omega t]\int_0^\infty dk \frac{k^2}{k^2-\frac{\omega^2}{c^2}}\frac{2i\sin kr}{ikr} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \exp[i\omega t]\int_0^\infty dk \frac{2k}{k^2-\frac{\omega^2}{c^2}}\frac{\sin kr}{r}$$

$$= \frac{1}{(2\pi)^3}\frac{1}{r}\int_{-\infty}^{\infty} d\omega \exp[i\omega t]\int_0^\infty dk \left(\frac{1}{k+\frac{\omega}{c}}+\frac{1}{k-\frac{\omega}{c}}\right)\sin kr$$

上半平面で右の積分路 $\oint_C dz\left(\frac{1}{z+\frac{\omega}{c}}+\frac{1}{z-\frac{\omega}{c}}\right)\exp[irz]=0$



$$\mathrm{P}\int_{-\infty}^{0} dx\left(\frac{1}{z+\frac{\omega}{c}}+\frac{1}{z-\frac{\omega}{c}}\right)\exp[irx]+\mathrm{P}\int_0^\infty dx\left(\frac{1}{z+\frac{\omega}{c}}+\frac{1}{z-\frac{\omega}{c}}\right)\exp[irx]+$$

$$+\lim_{\varepsilon\to 0}\int_\pi^0 i\varepsilon e^{i\theta}d\theta\left(\frac{1}{-\frac{\omega}{c}+\varepsilon e^{i\theta}+\frac{\omega}{c}}+\frac{1}{-\frac{\omega}{c}+\varepsilon e^{i\theta}-\frac{\omega}{c}}\right)\exp[ir(-\frac{\omega}{c}+\varepsilon e^{i\theta})]$$

$$+\lim_{\varepsilon\to 0}\int_\pi^0 i\varepsilon e^{i\theta}d\theta\left(\frac{1}{\frac{\omega}{c}+\varepsilon e^{i\theta}+\frac{\omega}{c}}+\frac{1}{\frac{\omega}{c}+\varepsilon e^{i\theta}-\frac{\omega}{c}}\right)\exp[ir(\frac{\omega}{c}+\varepsilon e^{i\theta})]$$

$$+\lim_{R\to\infty}\int_0^\pi iRe^{i\theta}d\theta\left(\frac{1}{Re^{i\theta}+\frac{\omega}{c}}+\frac{1}{Re^{i\theta}-\frac{\omega}{c}}\right)\exp[irRe^{i\theta}]=0$$

$$\mathrm{P}\int_\infty^0(-dy)\left(\frac{1}{-y+\frac{\omega}{c}}+\frac{1}{-y-\frac{\omega}{c}}\right)\exp[-iry]+\mathrm{P}\int_0^\infty dx\left(\frac{1}{x+\frac{\omega}{c}}+\frac{1}{x-\frac{\omega}{c}}\right)\exp[irx]-i\pi(\exp[-ir\frac{\omega}{c}]+\exp[ir\frac{\omega}{c}])=0$$

$$-\mathrm{P}\int_0^\infty dy\left(\frac{1}{y-\frac{\omega}{c}}+\frac{1}{y+\frac{\omega}{c}}\right)\exp[-iry]+\mathrm{P}\int_0^\infty dx\left(\frac{1}{x+\frac{\omega}{c}}+\frac{1}{x-\frac{\omega}{c}}\right)\exp[irx]=i2\pi\cos\frac{\omega}{c}r$$

$$\mathrm{P}\int_0^\infty dx\left(\frac{1}{x+\frac{\omega}{c}}+\frac{1}{x-\frac{\omega}{c}}\right)(\exp[irx]-\exp[-irx])=i2\pi\cos\frac{\omega}{c}r$$

$$\mathrm{P}\int_0^\infty dk\left(\frac{1}{k+\frac{\omega}{c}}+\frac{1}{k-\frac{\omega}{c}}\right)2i\sin rk = i2\pi\cos\frac{\omega}{c}r \quad \mathrm{P}\int_0^\infty dk\left(\frac{1}{k+\frac{\omega}{c}}+\frac{1}{k-\frac{\omega}{c}}\right)\sin kr = \pi\cos\frac{\omega}{c}r$$

$$\frac{1}{(2\pi)^3}\frac{1}{r}\int_{-\infty}^{\infty}d\omega\exp[i\omega t]\int_0^\infty dk\left(\frac{1}{k+\frac{\omega}{c}}+\frac{1}{k-\frac{\omega}{c}}\right)\sin kr$$

$$=\frac{1}{(2\pi)^3}\frac{1}{r}\int_{-\infty}^{\infty}d\omega\exp[i\omega t](\pi\cos\frac{\omega}{c}r)$$ $\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[i\omega t]d\omega$なし、$\omega=0$のとき $\frac{1}{4\pi}\frac{1}{r}$

$$=\frac{\pi}{2(2\pi)^3}\frac{1}{r}\int_{-\infty}^{\infty}d\omega\exp[i\omega t](\exp[i\frac{\omega}{c}r]+\exp[-i\frac{\omega}{c}r])$$

$$=\frac{1}{4(2\pi)^2}\frac{1}{r}\int_{-\infty}^{\infty}d\omega\left(\exp[i\omega(t+\frac{r}{c})]+\exp[i\omega(t-\frac{r}{c})]\right)$$

$$=\frac{1}{8\pi}\frac{1}{r}\left(\delta(t+\frac{r}{c})+\delta(t-\frac{r}{c})\right)=G(\boldsymbol{r},t) \quad r=|\boldsymbol{r}|$$

$$V(\boldsymbol{r},t)=\frac{1}{\varepsilon_0}\int G(\boldsymbol{r}-\boldsymbol{r}',t-t')\rho(\boldsymbol{r}',t')d\boldsymbol{r}'dt'$$

$$=\frac{1}{\varepsilon_0}\int\frac{1}{8\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left(\delta(t-t'+\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})+\delta(t-t'-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})\right)\rho(\boldsymbol{r}',t')d\boldsymbol{r}'dt'$$

$$=\frac{1}{8\pi\varepsilon_0}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left(\rho(\boldsymbol{r}',t+\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})+\rho(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})\right)d\boldsymbol{r}'$$ $c\to\infty$のとき $\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r},t) = -\delta(\boldsymbol{r})\delta(t)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int G(\boldsymbol{r},\omega)\exp[i\omega t]d\omega = -\delta(\boldsymbol{r})\int \exp[i\omega t]d\omega$$

$$\int\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\boldsymbol{r},\omega)\exp[i\omega t]d\omega = \int[-\delta(\boldsymbol{r})]\exp[i\omega t]d\omega$$

$$\int\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G(\boldsymbol{r},\omega)\exp[i\omega t]d\omega = \int[-\delta(\boldsymbol{r})]\exp[i\omega t]d\omega$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G(\boldsymbol{r},\omega) = -\delta(\boldsymbol{r})$$

$\delta(\boldsymbol{r})$は球対称　rにのみ依存する。$G(\boldsymbol{r},\omega)$もrにのみ依存する。

$$\frac{1}{r}\frac{d^2}{dr^2}(rG) + \frac{\omega^2}{c^2}G = -\delta(\boldsymbol{r})$$

$$\frac{d^2}{dr^2}(rG) + \frac{\omega^2}{c^2}(rG) = -r\delta(\boldsymbol{r})$$

$r \neq 0$**では**$\frac{d^2}{dr^2}(rG) + \frac{\omega^2}{c^2}(rG) = 0$**より**$rG = A\exp[i\frac{\omega}{c}t] + B\exp[-i\frac{\omega}{c}t]$　$G(\boldsymbol{r},\omega) = \dfrac{A\exp[i\frac{\omega}{c}t] + B\exp[-i\frac{\omega}{c}t]}{r}$

$c \to \infty$**では**$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G(\boldsymbol{r},\omega) \to \nabla^2 G(\boldsymbol{r},\omega) = -\delta(\boldsymbol{r})$**より**　$\lim_{c\to\infty} G(\boldsymbol{r},\omega) = \frac{1}{4\pi}\frac{1}{r}$

したがって、$A + B = \frac{1}{4\pi}$

$A = \frac{1}{4\pi}, B = 0$ **も** $A = 0, B = \frac{1}{4\pi}$ **も** $A = \frac{1}{8\pi}, B = \frac{1}{8\pi}$ **も解**