Applications of Computer Algebra – ACA 2021
Virtual. Online | July 23-27, 2021
Session on *“Algorithms for Polynomial System Solving and Their Applications’*

Double Ideal Quotient and Its Applications

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We introduce a special ideal operation “Double Ideal Quotient (DIQ)” and its variants, which are very useful tools for localizations of ideals. For ideals I and J in the n-variables polynomial ring $K[X] = K[x_1, \ldots, x_n]$ over a field K, we call $(I : (I : J))$ the *double ideal quotient* of I and J. Replacing such ideal quotients by saturations, we can consider its variants; $(I : (I : J)^\infty)$, $(I : (I : J^\infty)^\infty)$ and $(I : (I : J^\infty))$. DIQ and its variants have many properties on primary components and prime divisors; for example, they give us

- Criteria for prime divisors
- Criteria for primary components
- Criteria for isolated (embedded) prime divisors
- A way to compute equidimensional hulls
- A way to compute pseudo-primary components from given prime divisors
- A way to compute (isolated) primary components from given (isolated) prime divisors

We explain some details. First, we have the following criterion for prime divisors.

Proposition 1 ([5], Corollary 3.4)**.** *Let I be an ideal and P a prime ideal. Then, P is a prime divisor of I if and only if $P \supset (I : (I : P))$.*

Using a variant of DIQ, we have the following criterion for primary components.

Theorem 2 ([3], Theorem 26)**.** *Let I be an ideal and P a prime divisor of I. For a P-primary ideal Q, if $Q \not\supset (I : P^\infty)$, then the following conditions are equivalent.*

(A) *Q is a P-primary component for some primary decomposition of I.*

(B) *$(I : (I : J)^\infty) = J$ for $J = (I : P^\infty) \cap Q$.*

For a given prime divisor, we can check if it is isolated or embedded by the following criterion.

Proposition 3 ([3], Corollary 34)**.** *Let I be an ideal and P a prime divisor of I. Then,*

(i) *P is isolated if $(I:(I:P^\infty)^\infty) \neq K[X]$,*

(ii) *P is embedded if $(I:(I:P^\infty)^\infty) = K[X]$.*

Also, we can compute the *equidimensional hull* $\mathrm{hull}(I)$ of I, the intersection of all primary components of I whose dimension is that of I, by DIQ and a regular sequence as follows.

Proposition 4 ([5], Proposition 3.41)**.** *Let I be an ideal in $K[x_1,\ldots,x_n]$ and $u \subset I$ a regular sequence of length c, where c is the codimension of I i.e. $c = n - \dim(I)$. Then* $\mathrm{hull}(I) = (\langle u\rangle : (\langle u\rangle : I))$.

By combining equidimensional hull and DIQ, we can compute the isolated primary component directly from a given isolated prime divisor as follows.

Theorem 5 ([3], Theorem 36)**.** *Let I be an ideal and P an isolated prime divisor of I. Then*

$$\mathrm{hull}((I:(I:P^\infty)^\infty))$$

is the isolated P-primary component of I.

It is possible to compute embedded primary components from a given embedded prime divisors by the following proposition.

Proposition 6 ([1], Section 4)**.** *Let P be a prime divisor of I. For a sufficiently large integer m,* $\mathrm{hull}(I+P^m)$ *is a P-primary component of I.*

In the above proposition, we can check if m is large enough or not (i.e. $\mathrm{hull}(I+P^m)$ is a primary component or not) by using Theorem 2.

In the talk, we also see other applications of DIQ and its variants. Most of the propositions in the talk are introduced in references [2], [3] and [4].

Keywords
Gröbner basis, Ideal Operation, Localization, Primary Decomposition

References

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