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Noetherian representations for zero-dimensional ideals

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We introduce an effective algorithm for computing Noetherian differential operators of zero-dimensional primary ideals and present a new representation of a zero-dimensional ideal. We show that zero-dimensional ideals can be represented by the Noetherian (differential) operators and prime ideals. Moreover, we present new ideal computations as an application.

In 1938 W. Gröbner introduced differential operators to characterize membership in a polynomial ideal [7]. He derived such characterizations for prime or primary ideals and formulated the same problem [7] for any primary ideals.

Let x be an abbreviation of n variables $x_1, \ldots, x_n$, K a field with $\mathrm{char}(K) = 0$, $\partial_{x_i} := \frac{\partial}{\partial x_i}$ $(1 \leq i \leq n)$ and $D_X := K[x][\partial_{x_1}, \ldots, \partial_{x_n}]$ a ring of partial differential operators with coefficients in $K[x]$. If $I \subset K[x]$ is primary and $\sqrt{I} = P$ (the radical of I), then we say that I is P-primary.

L. Ehrenpreis and V. Palamodov gave the following theorem.

Theorem 1. *Let $Q \subset K[x]$ be a $\mathfrak{p}$-primary ideal. There exist partial differential operators $P_1, \ldots, P_\ell$ in D_X with the following property. A polynomial $h \in K[x]$ lies in the ideal Q if and only if $P_1(h), \ldots, P_\ell(h) \in \mathfrak{p}$.*

The partial differential operators $P_1, \ldots, P_\ell$ are called **Noetherian operators** for the primary ideal Q. These partial differential operators represent the difference of Q and the associated prime $\mathfrak{p}$, namely, the operators and the prime $\mathfrak{p}$ completly determine the structure of the primary ideal Q. Therefore, it is important to compute the operators for analyzing the primary ideal.

Recently, in the articles [2,3,4], Noetherian operators have been studied, and an algorithm for computing Noetherian operators have been given. Main tools of the articles are "Hilbert scheme" and "Macaulay's dual space".

In this talk, we only consider zero-dimensional ideals and give an algorithm for computing Noetherian operators of zero-dimensional primary ideals. The resulting algorithm is much faster than the algorithm given in [2,3,4], because the resulting algorithm mainly consists of linear algebra computations.

In order to construct the algorithm for computing Noetherian operators, we need the following propositions.

Proposition 1. *Let Q be a zero-dimensional primary ideal in $K[x]$ and $\sqrt{Q} = \mathfrak{p}$. Then, the set of Noetherian operators of Q in D_X is a finite dimensional vector space over the filed $K[x]/\mathfrak{p}$.*

Proposition 2. *Let I be a zero-dimensional ideal generated by $g_1, \dots, g_m$ in $K[x]$ and Q be a primary component of I with $\sqrt{Q} = \mathfrak{p}$. Let NB_Q be a basis of Noetherian operators of Q in D_X. Then, an arbitrary $P \in \mathrm{NB}_Q$ satisfies the following*
(i) $P(g_i) \in \mathfrak{p}$, $i = 1, \dots, m$.
(ii) The commutator $[P, x_i] := Px_i - x_iP \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_Q)$.

The outline of our algorithm for computing Noetherian operators is the following.

Outline of the algorithm

Input: $I \subset K[x]$: a zero-dimensional ideal.
Output: $\mathrm{Noether}(I) = \{(\mathfrak{p}_1, \mathrm{NB}_1), \dots, (\mathfrak{p}_\ell, \mathrm{NB}_\ell)\}$: $\mathfrak{p}_i$ is a associate prime of a primary component Q_i of I. NB_i is a basis of Noetherian operators of Q_i in D_X.

Step 1: Compute a prime decomposition of $\sqrt{I}$.
(Note that there exists an algorithm for computing a prime decomposition of $\sqrt{I}$. The algorithm is much faster than an algorithm for computing a primary decomposition of I. See [1].)
Step 2: (main part) For each prime ideal, compute Noetherian operators of the corresponding primary ideal. Repeat the following.
2-1: Take a candidate ∂^α of head terms.
2-2: Set $f = \partial^\alpha + \sum_{\partial^\alpha \succ \partial^\beta} h_\beta \partial^\beta$ where h_β is indeterminate.
2-3: Check Proposition 2 and decide h_β in $K[x]/\mathfrak{p}$.

Note that the input of the algorithm allows any zero-dimensional ideals.

The algorithm above has been implemented in the computed algebra system Risa/Asir.

We give an example.

Let $f = (x^2 + y^2)^2 + 3x^2y - y^3$, $g = x^2 + y^2 - 1$, and $I = \langle f, g \rangle \subset \mathbb{Q}[x, y]$. First, we compute a prime decomposition of $\sqrt{I}$, i.e.

$$\sqrt{I} = \langle x, y - 1 \rangle \cap \langle 4x^3 - 3, 2y + 1 \rangle.$$

Let $\mathfrak{p}_1 = \langle x, y - 1 \rangle$ and $\mathfrak{p}_2 = \langle 4x^3 - 3, 2y + 1 \rangle$. As I is a zero-dimensional ideal, I can be represented by

$$I = Q_1 \cap Q_2$$

where Q_1 is $\mathfrak{p}_1$-primary and Q_2 is $\mathfrak{p}_2$-primary. Our implementation outputs the following

$$\mathrm{Noether}(I) = \{(\mathfrak{p}_1, \{1, \partial_x\}),\ (\mathfrak{p}_2, \{1, \partial_x + 2x\partial_y\})\}$$

as Noetherian representation of the ideal I. The output $\{(\mathfrak{p}_1, \{1, \partial_x\}),\ (\mathfrak{p}_2, \{1, \partial_x + 2x\partial_y\})\}$ can be regared as a primary decomposition of I. Furthermore, we can regard $I = \langle f, g \rangle$ in the same light as $\{(\mathfrak{p}_1, \{1, \partial_x\}),\ (\mathfrak{p}_2, \{1, \partial_x + 2x\partial_y\})\}$.

In general, we call $\mathrm{Noether}(I) = \{(\mathfrak{p}_1, \mathrm{NB}_1), \ldots, (\mathfrak{p}_\ell, \mathrm{NB}_\ell)\}$ Noetherian representation of I.

As an application, we give an example of a sum of ideals. Let Q_1 and Q_2 be $\mathfrak{p}$-primary ideal in $\mathbb{Q}[x, y]$ where $\mathfrak{p} = \langle 4x^2 - 3, 2y + 1\rangle$. Assume that the Noetherian representations of Q_1 and Q_2 are given as

$$\mathrm{Noether}(Q_1) = \{(\mathfrak{p}, \{1, \partial_x, \partial_x^2 - 64x\partial_y\})\},\ \mathrm{Noether}(Q_2) = \{(\mathfrak{p}, \{1, \partial_x, \partial_y, \partial_x\partial_y\})\}.$$

Let's consider $Q_1 + Q_2$, the sum of the ideals. By computing the intersection of the sets of Noetherian operators, we can obtain $\mathrm{Noether}(Q_1 + Q_2)$, actually, since

$$\mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(1, \partial_x, \partial_x^2 - 64x\partial_y) \cap \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(1, \partial_x, \partial_y, \partial_x\partial_y) = \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(1, \partial_y),$$

we have,

$$\mathrm{Noether}(Q_1 + Q_2) = \{(\mathfrak{p}, \{1, \partial_y\})\}.$$

Base on the concept of Noetherian representations, we can construct a new framework for handing zero-dimensional ideals.

Keywords
Noetherian operators, primary ideals, zero-dimensional ideals

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