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Computing holonomic D-modules associated to a family of non-isolated hypersurface singularities via comprehensive Gröbner systems of PBW algebra

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We consider holonomic D-modules and microlocal b-functions associated to a family of non-isolated hypersurface singularities in the context of symbolic computation. We present an algorithm for computing them and describe a method for analyzing the structure of holonomic D-modules to compute microlocal b-functions. The key of the proposed method is the concept of local cohomology [1].

Let $f(x) \in K[x] = K[x_1, x_2, \cdots, x_n]$ be a polynomial of n variables with coefficients in a field K of characteristic zero. Let D denote the Weyl algebra:

$$D = K[x, \frac{\partial}{\partial x}] = K[x_1, x_2, \cdots, x_n, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}].$$

Let $D[s] = D \otimes_K K[s]$, where s is an indeterminate. A b-function, or Bernstein-Sato polynomial, is defined to be a monic generator of the ideal consisting of the polynomials $b(s)$ that satisfy $b(s)f^s = P(s, x, \frac{\partial}{\partial x})f^{s+1}$ for some partial differential operator $P \in D[s]$. For the case where the hypersurface $S = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ defined by f has non-isolated singularity, the (microlocal) b-functions and relevant holonomic D-modules are crucial in the study of singularity of the hypersurface S.

In this talk, we consider the case where the defining polynomial contains deformation parameters. More precisely, we consider the case

$$f_u(x) \in (K[u])[x] = K[u_1, u_2, \cdots, u_\ell, x_1, x_2, \cdots, x_n],$$

where $u = (u_1, u_n, \cdots, u_\ell)$ is regarded as a set of parameters. Based on our previous work [5] on comprehensive Gröbner basis in Poincaré-Birkhoff-Witt algebra, we extend the method given in [6] to parametric cases and present a new approach to studying deformation of non-isolated singularities.

In order to illustrate our approach, we will study and compute in particular the following examples.

Example 1 (D. B. Massey, 1990 [3])

$$f(x,y,z,w) = w^2 - yz^2 - xz^3 - z^4.$$

Example 2 (D. B. Massey, 1995 [4])

$$f(x,y,w_1,w_2,w_3) = y^2 - x^3 - (w_1^2 + w_2^2 + w_3^2)x^2.$$

Example 3 (D. B. Massey, 1995 [4])

$$f_u(x,y,z) = x^2 - y^3 - uzy^2$$

where u is a parameter.

Example 4 (J. Fernandez de Bobadilla, 2005 [1])

$$f_u(x_1,x_2,x_3,y_1,y_2) = x_3y_1^2 + 2x_2y_1y_2 + (ux_1 - x_3)y_2^2.$$

where u is a parameter.

Keywords
holonomic D-modules, non-isolated hypersurface singularities, comprehensive Gröbner systems, PBW algebra

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