On the classification problem of static pluriclosed metrics on minimal non-Kähler compact complex surfaces

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1 Introduction

It is well-known that the Ricci flow can be used to give an alternate proof of the uniformization theorem of Riemann surfaces:

Theorem 1.1. *Every compact connected Riemann surface is a quotient by a free, properly discontinuous action of a group on the unit disc, the complex plane, or the Riemann sphere. Moreover, it admits a Riemannian metric of constant scalar curvature.*

Likewise, we would like to find a geometric flow and to classify non-Kähler complex surfaces. Our flows should preserve Hermitianness, pluriclosedness and be close to the Kähler-Ricci flow. First of all, we recall the following fact:

Lemma 1.1. *Let $U \subset \mathbb{C}^n$ be an open subset homeomorphic to a ball, and suppose $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ is a pluriclosed form on U. There exists $\alpha \in \Lambda^{0,1}$ such that $\omega = \partial\alpha + \bar{\partial}\bar{\alpha}$.*

From this point of view, we define a flow of pluriclosed metrics using a d-closed $(1,1)$-form and a first-order $(0,1)$-form. Since we would like to reduce the flow to the Kähler-Ricci flow, we choose the Chern-Ricci curvature form as the closed form. In [21], Streets and Tian introduced a parabolic evolution equation of pluriclosed metrics with a pluriclosed initial metric ω_0 on a compact Hermitian manifold,

$$
\text{(PF)} \qquad \begin{cases} \frac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - \mathrm{Ric}(\omega(t)) =: -\Phi(\omega(t)), \\ \\ \omega(0) = \omega_0, \end{cases}
$$

which is called the pluriclosed flow (PF), where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are decompositions of the L^2-adjoint operator of the exterior differential operator with respect to metrics $g(t)$, $\mathrm{Ric}(\omega)$ is the second Chern-Ricci curvature of the Chern connection of a Hermitian metric ω; one of the Ricci-type curvatures of the Chern curvature. One has in complex coordinates the formula $\mathrm{Ric}_{i\bar{j}} = -g^{k\bar{l}}\partial_i\partial_{\bar{j}}g_{k\bar{l}} + g^{k\bar{l}}g^{r\bar{s}}\partial_i g_{k\bar{s}}\partial_{\bar{j}}g_{r\bar{l}} = -\partial_i\partial_{\bar{j}}\log\det(g)$. Note that the operator $\omega \mapsto \Phi(\omega)$ is a strictly elliptic operator for a pluriclosed metric ω, which means that the equation (PF) with a pluriclosed initial metric is a strictly parabolic equation. Hence the short-time existence and uniqueness of the solution (PF) follows from standard parabolic theory since the manifold is supposed to be compact. One can easily check that pluriclosed condition is preserved with using that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

Let (M^{2n}, g, J) be a complex manifold with complex structure J and compatible metric g. Let ω be the fundamental $(1,1)$-form of (M^{2n}, g, J) defined by $\omega(X, Y) = g(X, JY)$.

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Let ∇^L denote the Levi-Civita connection. Then the Chern connection ∇^C and the Bismut connection ∇^B are given by

$$g(\nabla^C_X Y, Z) = g(\nabla^L_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z), \quad g(\nabla^B_X Y, Z) = g(\nabla^L_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z)$$

respectively, where $d^c = i(\bar{\partial} - \partial)$, $d^c\omega(X, Y, Z) = -d\omega(JX, JY, JZ)$ for any $Y, Z \in T^{1,0}M$ and $X \in T^{\mathbb{C}}M$. Then we obtain $P^B(\omega) = P^C(\omega) - dd^*\omega$, where $P^C(\omega)$ is the Ricci form associated to the Chern connection of the metric g given by $P^C(\omega)(X, Y) = \frac{1}{2}\sum_{i=1}^{2n} g(\Omega^C(X, Y)e_i, Je_i)$, where $\{e_1, \ldots, e_{2n}\}$ is a local orthonormal frame of the tangent bundle TM and Ω^C is the curvature of the Chern connection such that $\Omega^C(X, Y) = [\nabla^C_X, \nabla^C_Y] - \nabla_{[X,Y]}$. The Ricci form $P^B(\omega)$ of the Bismut connection of the metric g is defined in a similar way (cf. [1]). This implies that a solution to (PF) can be written by

$$\frac{\partial}{\partial t}\omega(t) = -(P^B(\omega(t)))^{1,1},$$

where $(P^B(\omega(t)))^{1,1}$ denotes the projection of $P^B(\omega(t))$ onto $(1,1)$-forms. Streets and Tian showed that having a bound on the Bismut Ricci curvature $(P^B)^{1,1}$ suffices to obtain long-time existence for solutions of (PF).

Proposition 1.1. ([23, Theorem 1.2])Let $(M^{2n}, \omega(t), J)$ be a solution to (PF) starting at a pluriclosed metric on $[0, \tau)$. Suppose

$$\int_0^\tau \sup_{M\times\{t\}} |(P^B(\omega(t)))^{1,1}| dt < \infty.$$

Then the solution extends smoothly past time τ.

Let $(M^{2n}, \tilde{\omega}, J)$ be a compact complex manifold with pluriclosed metric, and let $\omega(t)$ be a solution to (PF) starting at a pluriclosed metric. We define a potential function $\varphi(t)$ along the solution $\omega(t)$ as follows:

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) - \Delta_{\omega(t)}\varphi(t) = \mathrm{tr}_{\omega(t)}\tilde{\omega} - n, \\ \varphi(0) = 0, \end{cases}$$

where $\Delta_{\omega(t)} = \mathrm{tr}_{\omega(t)}\partial\bar{\partial}$. It follows from standard parabolic theory that $\varphi(t)$ exists on the same time interval that $\omega(t)$ exists. We can actually define this potential function $\varphi(t)$ along the solution $\omega(t)$ with respect to a one-parameter family of pluriclosed metrics $\tilde{\omega}(t)$.

Proposition 1.2. ([23, Theorem 1.3])Let $(M, \tilde{g}, J)$ be a compact complex manifold and suppose $\omega(t)$ is a solution to (PF) starting at a pluriclosed metric on $[0, \tau)$ and suppose there is a constant C such that

$$\sup_{M\times[0,\tau)} |\varphi(t)| \leq C, \quad \sup_{M\times[0,\tau)} |T(\omega(t))|^2 \leq C,$$

where $T(\omega(t))$ is the torsion of the Chern connection associated to $\omega(t)$. Then $\omega(t) \to \omega(\tau)$ in C^∞, and the flow extends smoothly past time τ.

This proposition implies that the regularity requirement for solutions $\omega(t)$ to (PF) can be reduced to studying the behavior of the potential function φ and the torsion of solutions to (PF).

The $(1,1)$ Aeppli cohomology group of a compact complex manifold M of complex dimension n is the following;

$$\mathcal{H}^{1,1}_{\partial+\bar{\partial}}(M) = \frac{\{\mathrm{Ker}\partial\bar{\partial} : \Lambda^{1,1}_{\mathbb{R}}(M) \to \Lambda^{2,2}_{\mathbb{R}}(M)\}}{\{\partial\alpha + \bar{\partial}\bar{\alpha} | \alpha \in \Lambda^{0,1}(M)\}},$$

which is a finite dimensional space and isomorphic to the Bott-Chern cohomology group $H^{1,1}_{BC}(M,\mathbb{R}) = \{d\text{-closed real } (1,1)\text{-forms}\}/\sqrt{-1}\partial\bar{\partial}C^{\infty}_{\mathbb{R}}(M)$ for $n = 2$, where $\Lambda^{p,q}_{\mathbb{R}}(M)$, $C^{\infty}_{\mathbb{R}}(M)$ denote the space of smooth real (p,q)-forms and the space of smooth real functions on M respectively. Let the positive cone inside $\mathcal{H}^{1,1}_{\partial+\bar{\partial}}(M)$ be

$$\mathcal{P}_{\partial+\bar{\partial}}(M) = \{[\phi] \in \mathcal{H}^{1,1}_{\partial+\bar{\partial}}(M) | \exists\psi \in [\phi], \psi > 0\}.$$

It is obvous that a necessary condition for a solution to (PF) to exist is that the class $[\omega(t)] = [\omega_0] - tc_1^{BC} \in \mathcal{P}_{\partial+\bar{\partial}}(M)$, where c_1^{BC} denotes the first Bott-Chern class. Note that on a minimal non-Kähler compact complex surface M with pluriclosed metric ω_0, we have $[\omega_0] - tc_1^{BC}(M) \in \mathcal{P}_{\partial+\bar{\partial}}(M)$ for all $t \geq 0$. It is because that for a smooth real $\partial\bar{\partial}$-closed $(1,1)$-form ϕ, $[\phi] \in \mathcal{P}_{\partial+\bar{\partial}}(M)$ can be characterized by $\int_M \phi \wedge \gamma_0 > 0$ and $\int_D \phi > 0$ for every irreducible effective divisor with negative self intersection in the case of non-Kähler compact complex surfaces (cf. [23, Theorem 5.6]). Herre, γ_0 denotes a positive generator of $\frac{d\{\Lambda^1_{\mathbb{R}}\} \cap \Lambda^{1,1}_{\mathbb{R}}}{\sqrt{-1}\partial\bar{\partial}C^{\infty}_{\mathbb{R}}}$ which is identified with $\mathbb{R}$ via the L^2 inner product with a pluriclosed metric if $b_1 = $ odd, where $\Lambda^1_{\mathbb{R}}$ denotes the space of smooth real 1-forms.

Let M be a compact complex manifold of complex dimension n. We define the Kodaira dimension of M to be the infimum of $\kappa(M) \in [-\infty, \infty)$ such that there exists a constant C with $\dim H^0(M, K_M^{\otimes m}) \leq Cm^{\kappa(M)}$ for all large integer m with the convention that if $H^0(M, K_M^{\otimes m}) = \{0\}$, then we take $\kappa(M) = -\infty$, note that $\kappa(M)$ takes one of the values $-\infty, 0, 1, 2, \ldots, n$, where $K_M = \bigwedge^n T^*M$ is the canonical line bundle of M and $H^0(M, K_M^{\otimes m})$ is the vector space of global holomorphic sections of the holomorphic line bundle $K_M^{\otimes m}$. The Kodaira dimension $\kappa(M)$ measures the growth of the dimension of $H^0(M, K_M^{\otimes m})$ as $m \to \infty$. We say that M is of general type (K_M is big) if $\kappa(M) = n$.

Streets and Tian introduced the flow in order to study the toplology of class VII^+ surfaces. Class VII surfaces are compact complex surfaces with the Kodaira dimension $\kappa = -\infty$ and the first Betti number $b_1 = 1$. Class VII^+ surfaces are class VII surfaces with the second Betti number $b_2 > 0$. It is well known that the classification problem of class VII_0 surfaces (minimal class VII surfaces, "minimal" means no (-1)-curves) can be reduced into finding b_2- rational curves, thanks to Dloussky, Oeljeklaus and Toma:

Theorem 1.2. ([7] Dloussky, Oeljeklaus and Toma) Suppose that a class VII_0 surface S has $b_2(S)$-rational curves. Then S has a global spherical shell.

A spherical shell is an open surface which is biholomorphic to a neighborhood U of 3-sphere $S^3 \subset \mathbb{C}^2$. That a complex surface S has a global spherical shell (GSS) means that there is an open submanifold $V \subset S$ which is spherical shell and such that $S \setminus V$ is connected (cf. [10]). Class VII_0 surfaces which allows GSS's are well-understood. Streets and Tian conjectured that one can show the existance of sufficiently many rational curves with using (PF) and finish the classification of class VII_0 surfaces. In the case of $b_1 = 1$, they succeeded to classify under the following conjecture for non-Kähler surfaces:

Conjecture 1.1. ([23] Strong existence conjecture for non-Kähler surfaces) Let (M, g_0, J) be a non-Kähler compact complex surface with pluriclosed metric. Let $\omega(t)$ be the solution of the pluriclosed flow with initial condition g_0. Suppose $\omega(t)$ exists on $[0, \tau)$ and that

(1) $\lim_{t\to\tau} \mathrm{Vol}(g(t)) > 0$,

(2) There exists $\varepsilon > 0$ such that $\varepsilon < \int_D \omega(\tau) < \frac{1}{\varepsilon}$ for every irreducible effective divisor D with negative self intersection.

Then there exist uniform C^∞ estimates on $\omega(t)$ on $[0, \tau)$ depending on ε.

Theorem 1.3. ([23, Theorem 7.1]) Suppose Conjecture 1.1 holds. Then any class VII^+ surface contains an irreducible effective divisor of nonpositive self intersection.

It is well known that any divisor D on a class VII_0 has $D^2 \leq 0$. We write D^2 for its self intersection number.

We prove this result with using the following Lemma;

Lemma 1.2. ([23, Lemma 5.13]) Let (M^4, ω, J) be a compact complex surface with pluriclosed metric. Suppose the first Betti number $b_1 =$ odd and the Hodge number $h^{0,2} = 0$. Then

$$[\partial\omega] \neq 0 \in H^3(M, \mathbb{C}), \quad [\partial\omega] \neq 0 \in H^{2,1}(M),$$

where $H^{p,q}(M)$ is the Dolbeault cohomology group.

We define the Hodge number by $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M) = \dim_{\mathbb{C}} H^q(M, \Omega_M^p)$, where Ω_M^p is the sheaf of germs of holomorphic p-forms on a compact complex manifold M since we have Dolbeault's isomorphism $H^{p,q}(M) \cong H^q(M, \Omega_M^p)$.

We may apply this lemma to a class VII^+ surface and which leads a contradiction if we assme that there is no irreducible effective divisor with nonpositive self intersection. So we conclude that there exists an irreducible effective divisor D with $D^2 \leq 0$ on any class VII^+ surface. We now apply the result of Theorem 1.3 to the classification problem of class VII_0^+ surfaces. From Theorem 1.3, there always exists an irreducible effective divisor of nonpositive self intersection on any class VII_0^+ surface. It is well known that if D is an irreducible effective divisor on a class VII_0 surface, then D is either a nonsingular rational curve, a rational curve with a node or a nonsingular elliptic curve (cf. [20, (2.2) Lemma]). If D is an elliptic curve on a class VII_0 surface, then the surface is either an elliptic VII_0 surface (contains at least 3 elliptic curves), or a Hopf surface or a parabolic Inoue surface (cf. [20, (10.2) Theorem]). In our case of class VII_0^+, since we have $b_2 > 0$, it is restricted to be an elliptic VII_0 surface or a parabolic Inoue surface.

Here note that a class VII_0^+ surface S has at most $b_2(S)$-rational curves and if S admits a GSS, then there are exactly $b_2(S)$-rational curves. Thanks to Theorem 1.2, we see that Conjecture 1.1 implies that any class VII_0 surface with $b_2 = 1$ contains a GSS and can be classified into Kato surfaces (A Kato surface is a minimal compact complex surface with $b_2 > 0$ contains a GSS. It was shown by Kato that a Kato surface S is diffeomorphic to a Hopf surface blown up at $b_2(S)$-points and the fundamental group of S is isomorphic to $\mathbb{Z}$. A Kato surface does not admit Kähler metrics. cf. [10]) More precisely, it is an Enoki surface with $b_1 = 1$. An Enoki surface with $b_1 = 1$ is a class VII_0 surface with a rational curve C with a node with $C^2 = 0$. Note that it is known that an Enoki surface has a GSS (cf. [8]). This is the same result as Teleman showed:

Theorem 1.4. ([26] Teleman) Any class VII_0 surface with $b_2 = 1$ has a rational curve with a node.

A solution of the pluriclosed flow with pluriclosed initial condition ω_0 is equivalent to a solution of the following parabolic evolution equation on a compact complex manifold with pluriclosed metric, we here call it the Hermitian curvature flow (HCF):

$$\text{(HCF)} \qquad \begin{cases} \frac{\partial}{\partial t}\omega(t) = -S(\omega(t)) + Q^1(\omega(t)), \\ \\ \omega(0) = \omega_0, \end{cases}$$

where $S_{i\bar{j}}$ is the first Chern-Ricci tensor and Q^1 is a quadratic in the torsion of the Chern connection $Q^1_{i\bar{j}} = g^{k\bar{l}} g^{r\bar{s}} T_{ik\bar{s}} T_{\bar{j}\bar{l}r}$ [21, Proposition 3.3]. Note that $\mathrm{Ric} = S + \mathrm{div}^{\nabla} T - \nabla \bar{w}$ for any pluriclosed metric g, where T is the torsion of the Chern connection ∇ associated to g, $(\mathrm{div}^{\nabla} T)_{i\bar{j}} = g^{k\bar{l}} \nabla_{\bar{l}} T_{ki\bar{j}}$, $(\nabla \bar{w})_{i\bar{j}} = g^{k\bar{l}} \nabla_i T_{\bar{j}\bar{l}k}$. In the Kähler case, $S_{i\bar{j}} = \mathrm{Ric}_{i\bar{j}}$ is the Ricci curvature. Since one has in complex coordinates on a Hermitian manifold (M, g) the formula $S_{i\bar{j}} = -g^{k\bar{l}} \partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{k\bar{l}} g^{r\bar{s}} \partial_k g_{i\bar{s}} \partial_{\bar{l}} g_{r\bar{j}}$, we see that S is a strictly elliptic operator. Hence the equation $\frac{\partial}{\partial t}\omega = -S + Q^1$ is a strictly parabolic equation, and so the short-time existence and uniqueness of the solution to (HCF) with initial condition ω_0 follows from the standard theory. Since solutions to (HCF) are unique, the solution to (HCF) coincides with the solution to (PF) and the pluriclosed condition is preserved. If the initial condition ω_0 of the solution to HCF is Kähler, the solution is Kähler and HCF (equivalently (PF)) coincides with the Kähler-Ricci flow (cf. [21, Proposition 3.2], [22, Proposition 5.2]).

Definition 1.1. Let (M, J) be a complex manifold. A metric g is called a pluriclosed metric on M if g is a Hermitian metric whose associated real $(1,1)$-form (,which is called the fundamental $(1,1)$-form $\omega = g(J\cdot, \cdot)$ of the Hermitian metric g,) $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ satisfies $\partial\bar{\partial}\omega = 0$.

We also define the Gauduchon metric.

Definition 1.2. Let (M, J) be a complex manifold of dimension n. A metric g is called a Gauduchon metric on M if g is a Hermitian metric whose associated real $(1,1)$-form $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ satisfies $\partial\bar{\partial}(\omega^{n-1}) = 0$.

We will also refer to the associated real $(1,1)$-form ω as a pluriclosed metric or a Gauduchon metric. The following states that there are lot of Gauduchon metrics on any compact Hermitian manifold M.

Proposition 1.3. ([11] Gauduchon) Let M be a compact complex manifold of complex dimension n. Any Hermitian metric on M is conformally equivalent to a Gauduchon metric. If $n \geq 2$, then this Gauduchon metric is unique up to a positive factor.

Since all complex surfaces admit Hermitian metrics, there are always pluriclosed metrics (Gauduchon metrics) on compact complex surfaces.

The followings are the definition of the static pluriclosed metric and the degree of the pluriclosed metric.

Definition 1.3. Let (M, J) be a complex manifold with a pluriclosed metric ω on M. We say that ω is a static pluriclosed metric if for some constant λ,

$$\Phi(\omega) := -\partial\partial_g^*\omega - \bar{\partial}\bar{\partial}_g^*\omega + \mathrm{Ric}(\omega) = \lambda\omega.$$

Definition 1.4. Let (M^4, J) be a compact complex surface with a pluriclosed metric ω. Let

$$d_g := \deg_g(M) = \int_M \langle c_1^{BC}(M), \omega\rangle_g dV_g = \int_M c_1^{BC}(M) \wedge \omega = \int_M \mathrm{Ric}(\omega) \wedge \omega$$

denote the degree of the pluriclosed metric ω (cf. [21, Definition 3.7], [23, Definition 7.2]), where g is the metic associated to ω. Note that the value of degree does not depend on the representative of $c_1^{BC}(M)$ since the definition is made with respect to a fixed pluriclosed metric.

Note that if ω is Kähler and static, then it is Kähler-Einstein. On a complex surface, the existence of static pluriclosed metrics are closely related to the existence of algebraic and topological structures on the surface. For instance, the existence of static pluriclosed metric with $\lambda \neq 0$ implies the existence of a Hermitian-symplectic structure (Definition 1.4).

Streets and Tian studied static pluriclosed metrics on K3 surfaces (compact complex surfaces with the Hodge number $h^{0,1} = 0$ and the first Chern class $c_1 = 0$), 2-dimensional complex tori and complex surfaces of general type . On these surfaces, a static pluriclosed metric is Kähler-Einstein ([21, Proposition 5.6, 5.7]). Here we only introduce the case of surfaces of general type. We need the following proposition.

Proposition 1.4. ([21, Proposition 5.5])Let (M^4, J, g) be a compact complex surface with static pluriclosed metric. Then we have

$$c_1^{BC}(M)^2 - 2\lambda d_g + \frac{1}{2}d_g^2 \geq 0, \quad 2c_1^{BC}(M)^2 \leq d_g^2$$

with equality in either case if and only if either g is Kähler-Einstein or $c_1^{BC}(M) = 0$.

Then with using the proposition above, we obtain the result for surfaces of general type. Note that the field of meromorphic functions on a compact connected complex manifold is a finitely generated algebraic function field with a transcendency degree over $\mathbb{C}$, that does not exceed $\dim_{\mathbb{C}} M$ (cf. [2, *I*. (7.1) Theorem.]). This transcendency degree is called the algebraic dimension of M and denoted by $a(M)$. We always have

$$\kappa(M) \leq a(M) \leq \dim_{\mathbb{C}} M,$$

where $a(M) = \text{trans.deg}_{\mathbb{C}}\mathbb{C}(M)$, $\mathbb{C}(M)$ is the field of meromorphic functions. Here, $\text{trans.deg}_{\mathbb{C}}\mathbb{C}(M)$ denotes the maximum number of elements of the field $\mathbb{C}(M)$ algebraically independent over $\mathbb{C}$. We say elements $f_1, \ldots, f_n \in \mathbb{C}(M)$ are algebraically independent over $\mathbb{C}$ if there does not exist $0 \neq F \in \mathbb{C}[x_1, \ldots, x_n]$ such that $F(f_1, \ldots, f_n) = 0$. Note that if M is algebraic, then every meromorphic function is rational and the field of meromorphic functions is the field of rational functions on M.

Proposition 1.5. (cf. [21, Proposition 5.7]) Let (M^4, J, g) be a minimal compact complex surface of general type with static pluriclosed metric. Then g is Kähler-Einstein.

Proof. Let ω be a static pluriclosed metric $\Phi(\omega) = \lambda\omega$. Since M is of general type, i.e, the canonical bundle K_M is big, we have $\kappa(M) = 2$ and also we have $a(M) = 2$ from the inequality $\kappa(M) \leq a(M) \leq \dim_{\mathbb{C}} M$. Then M is algebraic (cf. [2, *IV*. (6.2) Theorem.]). (Indeed, a compact complex surface has algebraic dimension 2 if and only if it is projective (cf. [2, *IV*. (6.5) Corollary.]).) Hence M is embedded into a projective space $\mathbb{P}^N$ for some $N \in \mathbb{N}$. On the other hand, since the surface M is supposed to be minimal and $\kappa(M) \geq 0$, then K_M is nef (cf. [2, *III*. (2.4) Corollary.]). Since a nef line bundle L on a smooth projective variety is big if and only if $c_1(L)^2 > 0$, we obtain $c_1^{BC}(K_M)^2 > 0$. Since K_M is nef and big, we may conclude that the canonical bundle K_M is semi-ample. Therefore, there exists a well-defined holomorphic map induced by K_M, $\iota_s : M \to \mathbb{P}^N$, where $s = (s_0, s_1, \ldots, s_N)$ is an ordered basis of $H^0(M, K_M^{\otimes m})$ for some sufficiently large integer m. Then we obtain $c_1(K_M) = \frac{1}{m}[\iota_s^*\omega_{FS}]$, where ω_{FS} is the Fubini-Study metric on $\mathbb{P}^N$. Since M is a submanifold of $\mathbb{P}^N$, we may see that $\iota_s^*\omega_{FS}$ a Kähler metric on M, and hence we have $c_1(K_M) = \frac{1}{m}[\iota_s^*\omega_{FS}] > 0$ (which is equivalent to that K_M is ample from the Kodaira embedding theorem). The positivity of the canonical bundle K_M implies that we have $d_g < 0$. Since we have $0 < c_1^{BC}(M)^2 = \lambda d_g$, we then must have $\lambda < 0$. From Proposition 1.4, we have the inequality $2c_1^{BC}(M)^2 \leq d_g^2$ and then we obtain $d_g \leq -\sqrt{2c_1^{BC}(M)^2}$ and also $-\lambda \leq \sqrt{\frac{c_1^{BC}(M)^2}{2}}$. Combining these with the equality $2\int_X |\partial_g^*\omega|_g^2 dV_g = d_g - 2\lambda$, we obtain

$$2\int_X |\partial_g^*\omega|_g^2 dV_g \leq -\sqrt{2c_1^{BC}(M)^2} + 2\sqrt{\frac{c_1^{BC}(M)^2}{2}} = 0,$$

which implies that $\partial_g^*\omega = 0$, equivalent to that $\partial\omega = 0$. Therefore ω is Kähler. □

Thanks to the Kodaira-Enriques classification (cf. [2, pg.244]), we know that minimal non-Kähler compact complex surfaces fall into the following cases:

(1) (Primary and Secondary) Kodaira surfaces,

(2) Minimal properly elliptic surfaces,

(3) Class VII_0 surfaces with $b_2 = 0$ (These are classified into either Inoue surfaces or Hopf surfaces (cf. [25])),

(4) Class VII_0^+ surfaces

(a) $b_2 = 1$: These surfaces are classified into Kato surfaces with a rational curve C with a node with $C^2 = 0$ (cf. [26]),

(b) $b_2 = 2$: These surfaces have a cycle of rational curves and are classified into Enoki surfaces with $b_2 = 2$, parabolic Inoue surfaces with $b_2 = 2$, hyperbolic Inoue surfaces and besides, VII_0 surfaces contain a smooth rational curve and a rational curve with a node, all these surfaces are diffeomorphic to Kato surfaces (cf. [27]),

(c) $b_2 > 2$: These surfaces are unclassified.

A conjecture of Nakamura asserts that every class VII_0^+ surface should be a Kato surface. Kato surafces are only examples in class VII_0^+.

They have shown that Hopf surfaces admit a static pluriclosed metric with $\lambda = 0$. The following result implies $\lambda = 0$ on Hopf surfaces:

Proposition 1.6. ([21, Proposition 5.8]) Let (M^4, g, J) be a compact complex surface with static pluriclosed metric and suppose $\Sigma \subset M$ is a holomorphic curve such that $[\Sigma] = 0 \in H^2(M, \mathbb{R})$. Then $\lambda = 0$.

Since the Hopf surface is either an elliptic fibration over $\mathbb{P}^1$ or contains exactly two irreducible curves, there exists at least one holomorphic curve. This holomorphic curve is null-homologous since we have $H_2(S^3 \times S^1, \mathbb{R}) = 0$ because the Hopf surface is diffeomorphic to $S^3 \times S^1$. So if there exists a static pluriclosed metric, then we must have $\lambda = 0$ from Proposition 1.6. Indeed, the standard Hopf surface H admits a static pluriclosed metric with $\lambda = 0$ (cf. [21, Example 6.1]). We consider the standard Hermitian metric $\omega_H = \frac{\sqrt{-1}}{\rho^2}\partial\bar{\partial}\rho^2$, where $\rho^2 = \sum_{i=1}^{2} |z_i|^2$ for $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$, satisfies that on H,

$$\mathrm{Ric}(\omega_H) = -\sqrt{-1}\partial\bar{\partial}\log\frac{1}{r^4} = \frac{2}{\rho^2}\Big(\delta_{ij} - \frac{\bar{z}^i z^j}{\rho^2}\Big)\sqrt{-1}dz^i \wedge d\bar{z}^j \geq 0$$

since $\det(\omega_H) = \rho^{-4}$ and the eigenvalues of $\mathrm{Ric}(\omega_H)$ are $\lambda_1 = 0, \lambda_2 = \frac{2}{\rho^2}$. The metric ω_H is $\partial\bar{\partial}$-closed if its dimension is two, but it is not true in the higer dimensional cases. Indeed, in the 2-dimensional case, we have

$$\begin{aligned}
\partial\bar{\partial}\omega_H &= \partial\Big(-\frac{1}{\rho^4}(z^2 d\bar{z}^2 \wedge dz^1 \wedge d\bar{z}^1 + z^1 d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2)\Big) \\
&= \frac{2}{\rho^6}(z^1\bar{z}^1 + z^2\bar{z}^2)dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 - \frac{2}{\rho^4}dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 = 0.
\end{aligned}$$

So the metric ω_H satisfies the condition $\Lambda(\partial\bar{\partial}\omega_H) = 0$, equivalent to

$$\sum_k \Big(\frac{\partial(g_H)_{i\bar{j}}}{\partial z^k \partial\bar{z}^k} + \frac{\partial(g_H)_{k\bar{k}}}{\partial z^i \partial\bar{z}^j}\Big) = \sum_k \Big(\frac{\partial(g_H)_{i\bar{k}}}{\partial z^k \partial\bar{z}^j} + \frac{\partial(g_H)_{k\bar{j}}}{\partial z^i \partial\bar{z}^k}\Big)$$

for any i, j, and which also implies that the second Chern-Ricci curvature $\mathrm{Ric}(\omega_H)$ is nonnegative (cf. [18, Proposition 3.7]). Hence we have $c_1^{BC}(H) \geq 0$ and then $d_g \geq 0$.

Actually, the metric ω_H is a static pluriclosed metric with $\lambda = 0$ satisfying $\Phi(\omega_H) = 0$ since we have that $(S_{g_H})_{i\bar{j}} = (g_H)_{i\bar{j}}$ and $(Q^1_{g_H})_{i\bar{j}} = (g_H)_{i\bar{j}}$, where $(g_H)_{i\bar{j}}$ is the associated metric, S_{g_H} is the first Chern-Ricci curvature and $(Q^1_{g_H})_{i\bar{j}} = g_H^{k\bar{l}} g_H^{r\bar{s}} T^H_{ik\bar{s}} T^H_{\bar{j}\bar{l}r}$, T^H is the torsion of the Chern connection ∇^H associated to g_H. Hence we have $-S_{g_H} + Q^1_{g_H} = 0$. On the other hand, a Hopf surface blown up at $p > 0$ points admits no static pluriclosed metrics since then we must have $b_2 = p > 0$, which contradicts to $\lambda = 0$ (cf. [21, Proposition 5.9]).

Note that the Hopf manifold $S^{2n+1} \times S^1$ with the standard Hermitian metric has strictly positive first Chern-Ricci curvature and nonnegative second Chern-Ricci curvature, but it is not Kähler (cf. [18, Proposition 6.4]).

And also they showed that class VII^+ surfaces admit no static pluriclosed metrics:

Proposition 1.7. ([21, Proposition 6.3]) Class VII^+ surfaces admit no static pluriclosed metrics.

Proof. First we recall some structural facts for class VII surfaces:

$$h^{0,1} = 1, \quad h^{1,0} = h^{2,0} = h^{0,2} = 0, \quad (c_1^{BC})^2 = c_1^2 = -b_2.$$

Suppose that there exists a static pluriclosed metric ω satisfying $\Phi(\omega) = \lambda\omega$ for some constant λ on a class VII^+ surface M. Let $n := b_2(M) > 0$.

We compute by Stokes Theorem,

$$\lambda d_g = \lambda \int_M \omega \wedge c_1^{BC}(M) = \int_M \Phi(\omega) \wedge c_1^{BC}(M) = c_1^{BC}(M)^2 = -b_2(M) = -n < 0,$$

which implies that we must have $\lambda \neq 0$. Then by applying Proposition 1.8 and Proposition 3.2, M must be Kähler, contradiction.

□

The result in Proposition 1.7 can be said that it is the important fact for investigating the topology of class VII^+ surface. No static metric pluriclosed metrics on class VII^+ surfaces implies that (HCF) must encounter some non-trivial singularities on a class VII^+ surface. These singuralities should be closely related to curves on the surface. By the Kähler-Ricci flow, (-1)-curves are blown-down on a projective algebraic surface as finite time singularities, and all the (-2)-curves on a minimal projective surface of general type are contracted as infinite time singularities. As we confirmed, the classification problem of class VII^+ surfaces can be reduced to finding sufficiently many rational curves. One would be able to settle the problem by finding such curves as singularities of (HCF).

We study remained cases; Kodaira surfaces, minimal non-Kähler properly elliptic surfaces and Inoue surfaces.

Let us define the condition of a structure called Hermitian-symplectic.

Definition 1.5. Let (M, J) be a complex manifold of complex dimension n. A Hermitian-symplectic structure on M is a real 2-form $\tilde{\omega}$ such that $d\tilde{\omega} = 0$, and the projection of $\tilde{\omega}$ onto $(1,1)$-tensors determined by J is positive definite. We say that a complex manifold is Hermitian-symplectic if it admits a Hermitian-symplectic structure.

A Hermitian-symplectic structure is equivalent to ω being a taming form for the complex structure J. We say a symplectic form ω on a manifold tames an almost complex structure J if $\omega(X, JX) > 0$ for nonzero tangent vectors X.

It is well known that the space of symplectic manifolds is strictly larger than the space of Kähler manifolds. But we do not know whether the space of Hermitian-symplectic manifolds is strictly larger than the space of Kähler manifolds or not. However, for surfaces, we have the following result.

Proposition 1.8. (cf. [17, Theorem 1.2], [21, Proposition 1.6]) A complex surface is Hermitian-symplectic if and only if it is Kähler.

As we see above, the cases of Hopf surfaces and class VII_0^+ surfaces have already been investigated and classified. We study remained cases in the Kodaira-Enriques classification; primary and secondary Kodaira surfaces, minimal non-Kähler properly elliptic surfaces and three types of Inoue surfaces. Our main result is as follows:

Theorem 1.5. ([15, Theorem 1.1]) Kodaira surfaces, minimal non-Kähler properly elliptic surfaces and Inoue surfaces admit no static pluriclosed metrics.

The classification problem of static pluriclosed metrics on minimal non-Kähler compact complex surfaces is thus settled. Non-mimimal case is open as far as I know.

2 Preliminaries

Let (M, J) be a Hermitian manifold of complex dimension n with a Hermitian metric g on M. Let $\omega(u, v) = g(Ju, v)$ be the fundamental 2-form of g. Since g is J-invariant, $(2,0)$, $(0,2)$-components are vanished and ω is a smooth real $(1,1)$-form, which is called the fundamental $(1,1)$-form. In local coordinates, we have $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. On the Hermitian vector bundle $(T^{1,0}M, g)$, the Chern connection ∇ associated to g is the unique connection which is compatible with the Hermitian metric g and the complex structure J. Let Ω denote the curvature of the Chern connection ∇ and let $S_{i\bar{j}} = g^{k\bar{l}} \Omega_{k\bar{l}i\bar{j}}$ denote the first Chern-Ricci tensor, let $\mathrm{Ric}(\omega)_{i\bar{j}} = g^{k\bar{l}} \Omega_{i\bar{j}k\bar{l}}$ denote the second Chern-Ricci tensor.

On the holomorphic tangent bundle $T^{1,0}M$, there are three connections (cf. [18]);

(1) the complexified Levi-Civita connection ∇^L on $T^{1,0}M$,

(2) the Chern connection ∇^C on $T^{1,0}M$,

(3) the Bismut connection ∇^B on $T^{1,0}M$.

It is well known that if M is Kähler, then all three connections are the same.

The second Chern-Ricci curvature $\mathrm{Ric}(\omega)$ of ω, given locally by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \omega^n$, determines the Bott-Chern cohomology class denoted by $c_1^{BC}(M) \in H^{1,1}_{BC}(M, \mathbb{R})$, where

$$H^{1,1}_{BC}(M, \mathbb{R}) = \frac{\{\mathrm{Ker} d : \Lambda^{1,1}_{\mathbb{R}}(M) \to \Lambda^3_{\mathbb{R}}(M)\}}{\sqrt{-1}\partial\bar{\partial} C^\infty_{\mathbb{R}}(M)}.$$

. We call it the first Bott-Chern class of M. Note that we omit a factor of 2π that appears in the definition of $c_1^{BC}(M)$, and it is independent of the choice of Hermitian metrics.

Note that a compact complex manifold is said be in Fujiki's class $\mathcal{C}$ if it is bimeromorphic to a Kähler manifold. Class $\mathcal{C}$ includes all Moishezon manifolds since they are bimeromorphic to projective manifolds. If a compact complex manifold M is in $\mathcal{C}$, then the first Bott-Chern class $c_1^{BC}(M) = 0$ if and only if the first Chern class $c_1(M) = 0$ in $H^2(M, \mathbb{R})$ (cf. [28]). A compact complex manifold in the class $\mathcal{C}$ carries a Kähler current (cf. [6, Theorem 0.7]). It is well-known that every Moishezon surface is Kähler, but there are many non-Kähler Moishezon manifolds in higher dimension.

Although the second Chern-Ricci curvature represents the first Bott-Chern class, on a Hermitian manifold, it is possible that the first Chern-Ricci curvature is not in the same cohomology class as the second Chern-Ricci curvature. For instance, the Hopf surface $M \cong S^3 \times S^1$ with the standard Hermitian metric has strictly positive first Chern-Ricci curvature and nonnegative second Chern-Ricci curvature, but it is non-Kähler and it has $c_1(M) = 0$ in $H^2(M, \mathbb{R})$, $c_1^{BC}(M) \neq 0$ and $c_1^{BC}(M)^2 = c_1(M)^2 = -b_2(M) = 0$ (cf. [18, Proposition 6.4], [28, Example 3.3]).

Let $\Lambda^r(M) = \bigoplus_{p+q=r} \Lambda^{p,q}(M)$ for $0 \leq r \leq 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms. The exterior differential operator d decomposes into the operators ∂ and $\bar{\partial}$

$$\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M), \quad \bar{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M).$$

Let d_g^* denote the L^2-adjoint operator of d. Then d_g^* decomposes into ∂_g^* and $\bar{\partial}_g^*$

$$\partial_g^* : \Lambda^{p,q}(M) \to \Lambda^{p-1,q}(M), \quad \bar{\partial}_g^* : \Lambda^{p,q}(M) \to \Lambda^{p,q-1}(M).$$

We here again define a static pluriclosed metric on surfaces.

Definition 2.1. Let (M^4, J) be a compact complex surface with a pluriclosed metric ω on M. We say that ω is a static pluriclosed metric if ω satisfies $\Phi(\omega) = \lambda\omega$ for some constant λ, and $\mathrm{Vol}(g) = \int_M dV_g = 1$.

Definition 2.2. The Ricci curvatures are called negative (resp. nonnegative, positive, nonpositive) if the corresponding Hermitian matrices are negative (resp. nonnegative, positive, nonpositive).

3 Proof of Theorem 1.5

Let us consider a compact complex surface M. We rule out the scaling ambiguity by fixing the volume to be 1 as in Definition 2.1. Since then, for a static pluriclosed metric ω satisfying $\Phi(\omega) = \lambda\omega$ on M, we have $\int_M \omega \wedge \omega = \int_M \langle \omega, \omega \rangle_g dV_g = \int_M |\omega|_g^2 dV_g = 2$, where $|\omega|_g^2 = \langle \omega, \omega \rangle_g = g^{i\bar{j}} g^{k\bar{l}} g_{i\bar{l}} g_{k\bar{j}} = \sum_{k=1}^2 \delta_{kk} = 2$, and the volume form dV_g is written in local coordinates z^1, z^2 as $dV_g = \det g(\sqrt{-1})^2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2$, we can easily the following equality (cf. [21, Proposition 5.2])

$$(\dagger) \qquad 2\int_M |\partial_g^* \omega|_g^2 dV_g = d_g - 2\lambda.$$

Proof. (The equality $(\dagger)$))

We compute that

$$
\begin{aligned}
-2\lambda &= \int_M (-\Phi(\omega)) \wedge \omega \\
&= \int_M \left\langle \left(\partial \partial_g^* \omega + \bar{\partial} \bar{\partial}_g^* \omega - \mathrm{Ric}(\omega) \right), \omega \right\rangle_g dV_g \\
&= \int_M \left\langle \partial_g^* \omega, \partial_g^* \omega \right\rangle_g dV_g + \int_M \left\langle \bar{\partial}_g^* \omega, \bar{\partial}_g^* \omega \right\rangle_g dV_g - \int_M c_1^{BC}(M) \wedge \omega \\
&= \int_M |\partial_g^* \omega|_g^2 dV_g + \int_M |\bar{\partial}_g^* \omega|_g^2 dV_g - d_g.
\end{aligned}
$$

Since we have $|\partial_g^* \omega|_g^2 = |\bar{\partial}_g^* \omega|_g^2$, we obtain the equality (†). □

We claim the following.

Proposition 3.1. ([15, Proposition 2.1])Let M be a non-Kähler compact complex surface. If we have $d_g \leq 0$ for any pluriclosed metric ω, then there is no static pluriclosed metrics on M.

From this point of view, finding a Hermitian metric with non positive second Chern-Ricci curvature suffices to prove our main theorem for all three cases. For proving Proposition 3.1, we use the equality (†) and which leads to a contradiction to that non-Kähler under the assumption $d_g \leq 0$ for any pluriclosed metric ω. We will crucially use the fact that static pluriclosed metrics with nonzero constant λ automatically imply that there exists a Hermitian-symplectic structure (cf. [21, Proposition 5.10]) We need the following proposition to prove our claim above:

Proposition 3.2. ([21, Proposition 5.10]) Let (M, J) be a compact complex manifold with a static metric ω. If $\lambda \neq 0$, then M is a Hermitian-symplectic manifold.

Proof. (Proposition 3.1.) We assume that there exists a static pluriclosed metric ω on M, then we have from the equality (†),

$$
2 \int_M |\partial_g^* \omega|_g^2 dV_g = d_g - 2\lambda \leq -2\lambda
$$

since we have assumed that $d_g \leq 0$ for any pluriclosed metric ω.

If $\lambda \geq 0$, then we have $\int_M |\partial_g^* \omega|_g^2 dV_g = 0$ and $\partial_g^* \omega = 0$, equivalently we have $\partial \omega = 0$ since we have

$$
\int_M |\partial_g^* \omega|_g^2 dV_g = \int_M \langle \partial_g^* \omega, \partial_g^* \omega \rangle_g dV_g = \int_M \langle \partial \omega, \partial \omega \rangle_g dV_g = \int_M |\partial \omega|_g^2 dV_g.
$$

This means that ω is a Kähler metric on M, which contradicts with that M is non-Kähler. Hence we must have $\lambda < 0$. Since then we especially have $\lambda \neq 0$, we may apply Proposition 3.2, and hence M must be Hermitian-symplectic. From Proposition 1.8, then M must be Kähler, which is again a contradiction. Therefore, M admits no static pluriclosed metrics. □

First, let M be a Kodaira surface and ω be a pluriclosed metric on M. A Kodaira surface is a non-Kähler minimal compact complex surface with the Kodaira dimension $\kappa(M) = 0$, which can be classified into the following two cases: A primary Kodaira surface is a surface with $b_1(M) = 3$, admitting a holomorphic locally trivial fibration over an elliptic curve with an elliptic curve as typical fibre. A secondary Kodaira surface is a surface with $b_1(M) = 1$, admitting a primary Kodaira surface as unramified covering. These surfaces are elliptic fibre spaces over rational curves. In either case, since some power of the canonical bundle $K_M^{\otimes l} = lK_M$, for some $l \geq 1$ is holomorphically trivial (i.e., sometimes then K_M is called holomorphically torsion and satisfies $K_M^{\otimes l} \cong \mathcal{O}_M$, where $\mathcal{O}_M$ denotes the product trivial bundle), then we have $c_1^{BC}(K_M^{\otimes l}) = lc_1^{BC}(K_M) = 0$ and so $c_1^{BC}(M) = -c_1^{BC}(K_M) = 0$ (cf. [28, Proposition 1.1, Theorem 1.4], [31]). Hence there exists a Gauduchon metric ω_0 on M such that $\mathrm{Ric}(\omega_0) = 0$ (cf. [24], [29]). Then we have by Stokes Theorem, for any pluriclosed metric ω,

$$d_g = \int_M c_1^{BC}(M) \wedge \omega = \int_M \mathrm{Ric}(\omega) \wedge \omega = \int_M (\mathrm{Ric}(\omega) - \mathrm{Ric}(\omega_0)) \wedge \omega = \int_M \sqrt{-1}\partial\bar{\partial} f \wedge \omega = 0,$$

where f is a real-valued smooth function on M and we used that ω is pluriclosed by integrating by parts at the last equality. Since we have $d_g = 0$ for any pluriclosed metric ω on Kodaira surfaces M, there is no static pluriclosed metric on M by Proposition 3.1.

Second, let M be a non-Kähler properly elliptic surface. Here, non-Kähler means that M admits no Kähler metrics. A non-Kähler properly elliptic surface is a compact complex surface with its first Betti number $b_1(M) = $ odd and the Kodaira dimension $\kappa(M) = 1$ which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve S. The Kodaira-Enriques classification tells us that minimal non-Kähler properly elliptic surfaces are the only one case for minimal non-Kähler compact complex surfaces with the Kodaira dimension $\kappa = 1$.

We assume that M is minimal, that is, there is no (-1)-curve on M. It has been shown that the universal cover of M is $\mathbb{C} \times H$ (cf. [16, Theorem 28]), where H is the upper half plane in $\mathbb{C}$. Also, it is known that there exists a finite unramified covering $p : M' \to M$ with a covering transformation group $\Gamma(p) := \mathrm{Aut}(p)$, where $\mathrm{Aut}(p)$ is the set of automorphisms of p, i.e., any $\tau \in \mathrm{Aut}(p)$ is biholomorphic $\tau : M' \cong M'$, satisfies $p \circ \tau = p$ and is called a covering transformation. Here M' is a minimal properly elliptic surface, $\pi' : M' \to S'$ is an elliptic fiber bundle over a compact Riemann surface S' of genus at least 2, with fiber an elliptic curve E (since $\Gamma(p)$ acts also S', π' is $\Gamma(p)$-equivalent) (cf. [4, Lemma 1, 2]). The curve S' is a finite cover of S ramified at the images of the multiple fibers of π (precisely equal to the image of the quotient map $q : S' \to S$ of the set of finitely many fixed points under the $\Gamma(p)$-action), with quotient $S = S'/\Gamma(p)$, $\pi : M \to S$ is equal to the $\Gamma(p)$-quotient of $\pi' : M' \to S'$ and so that the map q satisfies $q \circ \pi' = \pi \circ p$ (cf. [14], [30]).

Note that when $\pi : M \to S$ is not a fiber bundle, π has no singular fibers, but it might have multiple fibers. Let $D \subset M$ be the set of all multiple fibers of π, so that $\pi(D)$ consists of finitely many orbifold points, which is precisely equal to the set of branch points, also equal to the image of the map q of fixed points under the $\Gamma(p)$-action on S'.

By considering the following holomorphic covering map

$$h : \mathbb{C} \times H \to \mathbb{C}^* \times H, \quad h(z, z') = (e^{-\frac{z}{2}}, z'),$$

we may work with $\mathbb{C}^* \times H$ instead of $\mathbb{C} \times H$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. We will write (z_1, z_2) for the coordinates on $\mathbb{C}^* \times H$ and $z_i = x_i + \sqrt{-1} y_i$, $x_i, y_i \in \mathbb{R}$ for $i = 1, 2$, which means that we have $y_2 > 0$.

It has been shown in [19] that there exists a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ with $H/\Gamma = S$, together with $\mu \in \mathbb{C}^*$ with $|\mu| \neq 1$ and $\mathbb{C}^*/\langle \mu \rangle = E$, and with a character $\chi : \Gamma \to \mathbb{C}^*$ such that M' is biholomorphic to the quotient of $\mathbb{C}^* \times H$ by the $\Gamma \times \mathbb{Z}$ -action defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n \right) \cdot (z_1, z_2) = \left((cz_2 + d) \cdot z_1 \cdot \mu^n \cdot \chi\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big), \frac{az_2 + b}{cz_2 + d} \right),$$

and the map $\pi' : M' \to S'$ is induced by the projection $\mathbb{C}^* \times H \to H$ (cf. [3, Proposition 2], [34, Theorem 7.4]). We define two forms on $\mathbb{C}^* \times H$ below:

$$\alpha := \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma := \sqrt{-1}(-\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2) \wedge (-\frac{2}{\bar{z}_1} d\bar{z}_1 - \frac{\sqrt{-1}}{y_2} d\bar{z}_2).$$

Note that we may work in a single compact fundamental domain for M' in $\mathbb{C}^* \times H$ using z_1, z_2 as local coordinates and we may assume that z_1, z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero.

Lemma 3.1. The forms α and γ are invariant under the $\Gamma \times \mathbb{Z}$ -action.

Proof. It suffices to show that the forms on $\mathbb{C}^* \times H$;

$$\frac{\sqrt{-1}}{y_2^2} dz_2 \wedge d\bar{z}_2, \quad -\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2$$

are $\Gamma \times \mathbb{Z}$ -invariant. For any $(A, n) \in \Gamma \times \mathbb{Z}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, we compute

$$\begin{aligned} \frac{\sqrt{-1}}{\mathrm{Im}\Big(\frac{az_2+b}{cz_2+d}\Big)} d\Big(\frac{az_2 + b}{cz_2 + d}\Big) \wedge d\Big(\frac{a\bar{z}_2 + b}{c\bar{z}_2 + d}\Big) &= \frac{\sqrt{-1}}{y_2^2} \frac{|cz_2 + d|^4}{(cz_2 + d)^2 (c\bar{z}_2 + d)^2} dz_2 \wedge d\bar{z}_2 \\ &= \frac{\sqrt{-1}}{y_2^2} dz_2 \wedge d\bar{z}_2 \end{aligned}$$

and

$$\begin{aligned} & -\frac{2}{(cz_2 + d) \cdot z_1 \cdot \mu^n \cdot \chi(A)} d((cz_2 + d) \cdot z_1 \cdot \mu^n \cdot \chi(A)) + \frac{\sqrt{-1}}{\mathrm{Im}\Big(\frac{az_2+b}{cz_2+d}\Big)} d\Big(\frac{az_2 + b}{cz_2 + d}\Big) \\ = & -\frac{2c}{cz_2 + d} dz_2 - \frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} \frac{|cz_2 + d|^2}{(cz_2 + d)^2} dz_2 \\ = & -\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} \Big(\frac{|cz_2 + d|^2 + \sqrt{-1} 2cy_2 (cz_2 + d)}{(cz_2 + d)^2} \Big) dz_2 \\ = & -\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2 \end{aligned}$$

Therefore, the forms α and γ are invariant under the $\Gamma \times \mathbb{Z}$ -action. □

It follows that they descend to M' and we may define a smooth strictly positive volume form Ω on M' by

$$\Omega = 2\alpha \wedge \gamma$$

since the volume form Ω is invariant under the $\Gamma \times \mathbb{Z}$ -action as we see in the following lemma.

Lemma 3.2. The volume form Ω is $\Gamma \times \mathbb{Z}$ -invariant and satisfies

$$\mathrm{Ric}(\Omega) = -\alpha \in c_1^{BC}(M') = -c_1^{BC}(K_{M'}),$$

where $K_{M'}$ is the canonical bundle over M'.

PROOF. Since the forms α and γ are $\Gamma \times \mathbb{Z}$ -invariant from Lemma 3.1, so is Ω.
We compute

$$\begin{aligned}
\mathrm{Ric}(\Omega) &= -\sqrt{-1}\partial\bar{\partial}\log\Omega \\
&= -\sqrt{-1}\partial\bar{\partial}\log\Big(\frac{4}{y_2^2|z_1|^2}\Big) \\
&= \sqrt{-1}2\partial\bar{\partial}\log y_2 \\
&= \sqrt{-1}2\partial\Big(\frac{\sqrt{-1}}{2y_2}\Big)d\bar{z}_2 \\
&= \sqrt{-1}\Big(-\frac{\sqrt{-1}}{y_2^2}\Big)\Big(-\frac{\sqrt{-1}}{2}\Big)dz_2 \wedge d\bar{z}_2 \\
&= -\frac{\sqrt{-1}}{2y_2^2}dz_2 \wedge d\bar{z}_2 = -\alpha.
\end{aligned}$$

□

The form α induces a unique Kähler-Einstein metric $\omega_{S'}$ with $\mathrm{Ric}(\omega_{S'}) = -\omega_{S'}$ on S'. From Lemma 3.2, we obtain that

$$\mathrm{Ric}(\Omega) = -\pi'^*\omega_{S'} = \pi'^*\,\mathrm{Ric}(\omega_{S'}).$$

It follows that we have that

$$c_1^{BC}(M') = \pi'^* c_1(S').$$

Since the genus of S' is at least 2, we have $c_1(S') < 0$. Therefore we have $c_1^{BC}(M') \le 0$, which implies that the canonical bundle $K_{M'}$ is nef. We say that a holomorphic line bundle L over a compact complex surface N is nef if we have $\int_C c_1^{BC}(L) \ge 0$ for all curves C in N. If C is not smooth, then we integrate over C_{reg}, the set of points $p \in C$ for which C is a submanifold of N near p, since Stokes' Theorem still holds for C_{reg} (cf. [12, p.33]).

Lemma 3.3. If the canonical bundle $K_{M'}$ is nef, then the canonical bundle K_M is nef.

PROOF. Since $p : M' \to M$ is as unramified finite covering, i.e., for a sufficiently small open set $U \subset M$ we have that $p^{-1}(U)$ is a disjoint union of finitly many copies U_j of U

and then $p : U_j \to U$ is biholomorphic for each j, we may compute that for any Hermitian metric ω and any curve C on M,

$$\int_C (-\operatorname{Ric}(\omega)) = \int_{p^*C} (-p^* \operatorname{Ric}(\omega)) = \int_{p^*C} (-\operatorname{Ric}(p^*\omega)) = \int_{p^*C} c_1^{BC}(K_{M'}) \geq 0$$

since K_M' is assumed to be nef, where $p^*\omega$ is Hermitian and p^*C is a curve on M'. Hence K_M is also nef. □

If we have $c_1^{BC}(K_M) = -c_1^{BC}(M) < 0$, for any Hermitian metric ω_M on M, there exists a real smooth function F on M such that $-\operatorname{Ric}(\omega_M) + \sqrt{-1}\partial\bar{\partial}F < 0$ and then for any curve C in M, by Stokes Theorem,

$$\int_C c_1^{BC}(K_M) = \int_C (-\operatorname{Ric}(\omega_M)) = \int_C (-\operatorname{Ric}(\omega_M) + \sqrt{-1}\partial\bar{\partial}F) < 0,$$

which contradicts to the result that K_M is nef in Lemma 3.3. Hence, that $c_1^{BC}(M') \leq 0$ gives us that $c_1^{BC}(M) \leq 0$. With using this result, we obtain that $d_g \leq 0$ for any pluriclosed metric ω. By Proposition 3.1, there is no static pluriclosed metric on M.

Finally, we study a static pluriclosed metric on Inoue surfaces. Inoue surfaces form three families, S_M, $S^+_{N,p,q,r;\mathbf{t}}$ and $S^-_{N,p,q,r}$ (cf. [13], [30]). First of all, we construct the Inoue surface of type S_M. Let $M \in \mathrm{SL}(3,\mathbb{Z})$ be a matrix with one real eigenvalue $\tau > 1$ and two complex conjugate eigenvalues $\eta \neq \bar{\eta}$. Let (l_1, l_2, l_3) be a real eigenvector for M with eigenvalue τ and (m_1, m_2, m_3) be an eigenvector with eigenvalue η. We write $z_1 = x_1 + \sqrt{-1}y_1$ for the coordinate on $\mathbb{C}$ and $z_2 = x_2 + \sqrt{-1}y_2$ for the coordinate on $H = \{y_2 > 0\}$, the upper half plane in $\mathbb{C}$. Let G_M be the group of automorphisms of $\mathbb{C} \times H$, which is generated by

$$f_0(z_1, z_2) = (\eta z_1, \tau z_2), \quad f_i(z_1, z_2) = (z_1 + m_i, z_2 + l_i)$$

for $i = 1, 2, 3$, $(z_1, z_2) \in \mathbb{C} \times H$, $1 \leq i \leq 3$. We define S_M to be the quotient surface $(\mathbb{C} \times H)/G_M$, which is a T^3-torus bundle over a circle.

We consider the subgroup $\tilde{G}_M \subset G_M$ generated by f_1, f_2 and f_3, which is isomorphic to $\mathbb{Z}^3$ and acts on $\mathbb{C} \times H$ properly discontinuously and freely, with quotient the product $T^3 \times \mathbb{R}_{>0}$. If Γ is a discrete group and X is a Hausdorff topological space such that Γ acts on X, we say that this action is properly discontinuously if given two points $x, y \in X$, there are open neighborhoods U_x of x and U_y of y for which $(\gamma U_x) \cap U_y \neq \emptyset$ for only finitely many $\gamma \in \Gamma$, equivalent to that X/Γ is Hausdorff (cf. [5]). We say that a toplogical transformation group Γ acts on a topological space X freely if Γ satisfies that if $\gamma x = x$ for some $x \in X$, then $\gamma = e$, where $e \in \Gamma$ is unit.

The projection $\pi : T^3 \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is induced by $(z_1, z_2) \mapsto \mathrm{Im} z_2$ for $(z_1, z_2) \in \mathbb{C} \times H$. Since f_0 descends to a map $T^3 \times \mathbb{R}_{>0} \to T^3 \times \mathbb{R}_{>0}$, we obtain that

$$S_M = (T^3 \times \mathbb{R}_{>0})/\langle f_0 \rangle,$$

and since $\tau \in \mathbb{R}_{>1}$, f_0 maps $T_y = \pi^{-1}(y)$ to $T_{\tau y} = \pi^{-1}(\tau y)$. Especially, we have a diffeomorphism $F_0 : T_1 \to T_\tau$ induced by f_0. Then we have that S_M is diffeomorphic to the quotient space $(T^3 \times [1, \tau])/\sim$, where $(p, 1) \sim (F_0(p), \tau)$.

We may assume that z_1 and z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero since we may work in a single compact fundamental domain for S_M in $\mathbb{C} \times H$.

On $\mathbb{C} \times H$, we define nonnegative $(1,1)$-forms α' and β by

$$\alpha' := \frac{\sqrt{-1}}{4y_2^2} dz_2 \wedge d\bar{z}_2, \quad \beta := \sqrt{-1} y_2 dz_1 \wedge d\bar{z}_1.$$

Since these forms are invariant under the G_M -action (note that $\tau|\eta|^2 = 1$), they descend to S_M. We then can define a Hermitian metric on S_M, so called the Tricerri metric ω_T (cf. [9, Section 2], [30, Section 5] and [32]) by $\omega_T := 4\alpha' + \beta$, which is pluriclosed on S_M. Then we have

$$\mathrm{Ric}(\omega_T) = -\sqrt{-1}\partial\bar{\partial} \log \omega_T^2 = \sqrt{-1}\partial\bar{\partial} \log y_2 = -\alpha' \leq 0$$

and $c_1^{BC}(S_M) \leq 0$, which implies that we have $d_g \leq 0$ for any pluriclosed metric ω.

We next construct the Inoue surface of type $S^+_{N,p,q,r;\mathbf{t}}$. Let $N = (n_{ij}) \in \mathrm{SL}(2,\mathbb{Z})$ with two real eigenvalues $\tau > 1$ and $\frac{1}{\tau}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues τ and $\frac{1}{\tau}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$ and a complex number $\mathbf{t} \in \mathbb{C}$. Define $e_i := \frac{1}{2} n_{i1}(n_{i1} - 1)a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1)a_2 b_2 + n_{i1} n_{i2} b_1 a_2$ for $i = 1, 2$. Using N, a_i, b_i, p, q, r, one gets two real numbers (c_1, c_2) as solutions of the linear equation $(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q)$. Let G^+_N be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (z_1 + \mathbf{t}, \tau z_2), \;\; f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i), \;\; f_3(z_1, z_2) = (z_1 + \frac{b_1 a_2 - b_2 a_1}{r}, z_2)$$

for $i = 1, 2$, $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^+_{N,p,q,r;\mathbf{t}}$ to be the quotient surface $(\mathbb{C} \times H)/G^+_N$, which is diffeomorphic to a bundle over a circle with fiber a compact 3-manifold X.

We consider the subgroup $\tilde{G}^+_N \subset G^+_N$ generated by f_1, f_2 and f_3. Write $z_i = x_i + \sqrt{-1} y_i$ for $i = 1, 2$. For fixed $y_2 = \mathrm{Im} z_2$, the group $\tilde{G}^+_N$ acts on $\{(x_2, y_2, z_1) | x_2 \in \mathbb{R}, z_1 \in \mathbb{C}\} \cong \mathbb{R}^3$ properly discontinuously and freely, with quotient a compact 3-manifold X_{y_2}. Compact 3-manifolds X_y for different values of y are all diffeomorphic to a fixed compact 3-manifold X. We may consider that the group $\tilde{G}^+_N$ acts on $\mathbb{C} \times H$ with the quotient diffeomorphic to the product $X \times \mathbb{R}_{>0}$ with the projection $\pi : X \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ induced by $(z_1, z_2) \mapsto y_2$ and with $X_{y_2} = \pi^{-1}(y_2)$. Since f_0 descends to a map $X \times \mathbb{R}_{>0} \to X \times \mathbb{R}_{>0}$, we have that

$$S^+_{N,p,q,r;\mathbf{t}} = (X \times \mathbb{R}_{>0})/\langle f_0 \rangle.$$

Since $\tau \in \mathbb{R}_{>1}$, f_0 maps X_1 to X_τ and then induces a diffeomorphism F_0 of X such that $S^+_{N,p,q,r;\mathbf{t}}$ is diffeomorphic to the quotient space $(X \times [1, \tau])/\sim$, where $(p, 1) \sim (F_0(p), \tau)$.

We finally construct the Inoue surface of type $S^-_{N,p,q,r}$. Let $N = (n_{ij}) \in \mathrm{GL}(2,\mathbb{Z})$ with $\det N = -1$ and with two real eigenvalues $\tau > 1$ and $-\frac{1}{\tau}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues τ and $-\frac{1}{\tau}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$. One gets two real numbers (c_1, c_2) as solutions of the following linear equation $-(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q)$, where e_i for each $i = 1, 2$ is defined as in the case $S^+_{N,p,q,r;\mathbf{t}}$. Let G^-_N be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (-z_1, \tau z_2), \;\; f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i), \;\; f_3(z_1, z_2) = (z_1 + \frac{b_1 a_2 - b_2 a_1}{r}, z_2)$$

for $i = 1, 2$ and for $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^-_{N,p,q,r}$ to be the quotient surface $(\mathbb{C} \times H)/G^-_N$. Note that every surface $S^-_{N,p,q,r}$ has as an unramified double cover an Inoue surface $S^+_{N^2,p',q',r;0}$ for suitable integers p', q'. In fact, we have the involution of $S^+_{N^2,p',q',r;0}$: $\iota(z_1, z_2) = (-z_1, \tau z_2)$ satisfies $\iota^2 = \mathrm{Id}$ and $S^-_{N,p,q,r} = S^+_{N^2,p',q',r;0}/\iota$.

As in the case of the surface S_M, we may assume that local holomorphic coordinates z_1 and z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero.

Since $\tau > 1$, we may write $\mathrm{Im}\mathbf{t} = m \log \tau$ for some $m \in \mathbb{R}$. Note that $\mathbf{t}$ is real if and only if $m = 0$. We define $(1,1)$-forms α'' and γ' on $\mathbb{C} \times H$ by

$$\alpha'' := \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma' := \sqrt{-1}\Big(dz_1 - \frac{y_1 - m\log y_2}{y_2} dz_2\Big) \wedge \Big(d\bar{z}_1 - \frac{y_1 - m\log y_2}{y_2} d\bar{z}_2\Big).$$

Since these forms are invariant under the G^+_N -action, they descend to $S^+_{N,p,q,r;\mathbf{t}}$.

Then we can define the Vaisman metric on $S^+_{N,p,q,r;\mathbf{t}}$ $\omega_V := 2\alpha'' + \gamma'$, which is a pluriclosed metric on $S^+_{N,p,q,r;\mathbf{t}}$ (cf. [9, Section 3], [30, Section 6] and [33]). This metric satisfies on $S^+_{N,p,q,r;\mathbf{t}}$,

$$\mathrm{Ric}(\omega_V) = -\sqrt{-1}\partial\bar{\partial}\log\omega_V^2 = \sqrt{-1}\partial\bar{\partial}\log y_2^2 = -\alpha'' \leq 0.$$

Hence we obtain that $c_1^{BC}(S^+) \leq 0$ and that $d_g \leq 0$ for any pluriclosed metric ω on $S^+_{N,p,q,r;\mathbf{t}}$.

In the case of $S^-_{N,p,q,r}$ for a matrix $N \in \mathrm{GL}(2, \mathbb{Z})$ with $\det N = -1$ and arbitrary fixed $p, q, r \in \mathbb{Z}$ with $r \neq 0$, we concider forms on $\mathbb{C} \times H$

$$\alpha'' = \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma'^- := \sqrt{-1}\Big(dz_1 - \frac{y_1}{y_2} dz_2\Big) \wedge \Big(d\bar{z}_1 - \frac{y_1}{y_2} d\bar{z}_2\Big),$$

which are invariant under the G^-_N -action and so they descend to $S^-_{N,p,q,r}$. Hence we can define a Hermitian metric on $S^-_{N,p,q,r}$, $\omega_V^- := 2\alpha'' + \gamma'^-$, which is pluriclosed.

Denote $\sigma : S^+_{N^2,p',q',r;0} \to S^-_{N,p,q,r}$ the quotient map for suitable $p', q' \in \mathbb{Z}$, which is an unramified double covering. We pull back the metric ω_V^- via σ to $S^+_{N^2,p',q',r;0}$. The metric $\sigma^*\omega_V^-$ coincides with the metric $(\omega_V)_0$ which is the metric ω_V with $m = 0$. Then we have

$$\sigma^* \mathrm{Ric}(\omega_V^-) = \mathrm{Ric}(\sigma^*\omega_V^-) = \mathrm{Ric}((\omega_V)_0) = -\alpha'' \leq 0.$$

Since α'' is invariant under the $G^+_{N^2}$ and G^-_N -actions, we obtain on $S^-_{N,p,q,r}$,

$$\mathrm{Ric}(\omega_V^-) = -\alpha'' \leq 0.$$

Hence we obtain that $c_1^{BC}(S^-) \leq 0$ and this implies that we have $d_g \leq 0$ for any pluriclosed metric ω on $S^-_{N,p,q,r}$. By applying Proposition 3.1, we conclude that Inoue surafces of all types S_M, S^+ and S^- admit no static pluriclosed metrics.

In all cases; Kodaira surfaces, minimal non-Kähler properly elliptic surfaces and Inoue surfaces of all three types, we can obtain $d_g \leq 0$ for any pluriclosed metric ω and hence we conclude that these surfaces admit no static pluriclosed metrics.

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