On bifurcation and local rigidity of triply periodic minimal surfaces in the three-dimensional Euclidean space

(Joint work with T. Shoda and P. Piccione)

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Geometric Analysis in Geometry and Topology 2015
(Tokyo U. of Science, November 11, 2015)
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References
1 Introduction

Object: orientable connected embedded triply-periodic minimal surfaces (TPMS’s) in $\mathbb{R}^3$. (= cpt. minimal surfaces in flat $T^3$.)

[The most well-known examples of TPMS’s]

Schwarz P surface (19c)  
Schwarz D surface (19c)

(http://www.indiana.edu/ minimal/archive/Triply/genus3.html)

Alan Schoen’s Gyroid (1970)  
one period of D surface
\( \text{TPMS}(\mathbb{R}^3) := \{ \text{orientable connected embedded \textbf{triply-periodic} minimal surfaces (TPMS’s) in } \mathbb{R}^3 \} \)

\( \uparrow \)

\( \text{CMS}(T^3) := \{ \text{orientable connected embedded \textbf{compact} minimal surfaces in flat } T^3 \}. \) \( (g := \text{genus of the considered surface}) \)

- \( g = 0: \exists \) \( \text{ Gauss-Bonnet Th. } \)
- \( g = 1: \text{ Totally geodesic subtorus } T^2 \leftrightarrow \text{ planes in } \mathbb{R}^3 \)
- \( g = 2: \exists \) \( \text{ Gauss-Bonnet } + \text{ Gauss map is anti-holo. to } S^2 \)
- \( g \geq 3: \text{ There are many examples.} \)

\( \circ \) Classification is difficult.

\( \circ \) We study local structures of \( \text{TPMS}(\mathbb{R}^3) \).

Remark: TPMS’s also interest physicists and chemists because they appear in various natural phenomenon: Self-assembly of diblock copolymers in soft matter physics, \( \cdots \)
Main results (roughly):

(A) For each “generic” \( M_0 \in \text{TPMS}(\mathbb{R}^3) \), \( \exists \Omega : \text{neighborhood of } M_0 \text{ s.t. } \Omega \cap \text{TPMS}(\mathbb{R}^3) \) is 5-dimensional space (up to homothety and congruence in \( \mathbb{R}^3 \)). “5-dimension” corresponds to the space of all lattices in \( \mathbb{R}^3 \).

Examples of “generic” TPMS’s:

Strictly stable TPMS. = The second variation of area is positive for all “volume-preserving” variations. Ex: Schwarz P surface, Schwarz D surface, Alan Schoen’s Gyroid.

(B’) There are singularities in TPMS(\( \mathbb{R}^3 \)).
2 Definitions and main theorems

\( \Sigma \): 2-dim. oriented compact conn. \( C^\infty \) manifold with \( g(\Sigma) \geq 3 \),
\( X : \Sigma \rightarrow T^3_\Lambda := \mathbb{R}^3/\Lambda , \) minimal immersion into \( T^3_\Lambda = (T^3, g_\Lambda) \),
\( J[\varphi] := \Delta \varphi - 2K\varphi, \quad K \) is the Gauss curvature of \( X \).

\( J \) is the Jacobi operator of \( X \). \( H : \) mean curvature of surface.

For a variation \( X_\epsilon = X + \epsilon (\varphi \tilde{n} + \xi) + O(\epsilon^2) \) of \( X \), \( J[\varphi] = 2 \delta H \).

Consider eigenvalue problem: \((*) \) \( J[\varphi] = -\lambda \varphi, \quad \varphi \in C^{2,\alpha}(\Sigma) - \{0\} \).

Denote by \( \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \) the eigenvalues of \((*)\).

**Index of** \( X \): \( \text{Ind}(X) := \#\{j \mid \lambda_j < 0\} \)
= \( \text{dim}\{\text{variation vector fields which diminishes area}\} \),

**Nullity of** \( X \): \( \text{Nul}(X) := \#\{j \mid \lambda_j = 0\} \).

**Remark.** \( \text{Ind}(X) \geq 1. \) \((\leftarrow X_\epsilon = X + \epsilon \tilde{n} : \text{parallel surfaces.})\)
\( \text{Nul}(X) \geq 3. \) \((\leftarrow X_\epsilon = X + \epsilon e_i, \text{ where } \{e_1, e_2, e_3\} \text{ is a basis in } \mathbb{R}^3.\)
Notations:

Denote by $\mathcal{T}(T^3)$ the set of all flat metrics in $T^3$ (modulo isometry), and by $[\ ]$ the isometry class.

Let $\Lambda_0$ be a lattice in $\mathbb{R}^3$. Let $X_0 : \Sigma \to T^3_{\Lambda_0}$ be a minimal embedding.

For any $[\Lambda]$ close to $[\Lambda_0]$, and $\varphi \in C^{2,\alpha}(\Sigma)$ close to 0, we define an embedding $X_{\varphi,\Lambda} : \Sigma \to T^3$ as

$$X_{\varphi,\Lambda}(p) = \exp_{X_0(p)}^{g_\Lambda} (\varphi(p) \cdot \tilde{n}_{X_0(p)}^{g_\Lambda}), \quad p \in \Sigma,$$

where $\exp^{g_\Lambda}$ is the exponential map, and $\tilde{n}_{X_0}^{g_\Lambda}$ is the unit normal vector field along $X_0$ in $(T^3, g_\Lambda)$. All minimal embeddings near $X_0$ can be represented in this form.

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Recall $X_{\varphi,\Lambda}(p) = \exp^{g_{X_0(p)}}(\varphi(p) \cdot \vec{n}_{X_0(p)})$, $p \in \Sigma$.

Theorem A (Rigidity. Meeks(1990)[6] for special cases. Ejiri[1], K-P-S[5]). Let $X_0 : \Sigma \to T^3_{\Lambda_0}$ be a compact minimal embedding with $g(\Sigma) \geq 3$ and $\text{Nul}(X_0) = 3$. Then,

$\exists V :$ a neighborhood of $[\Lambda_0]$

in $T(T^3) = \{\text{flat metrics on } T^3\}/\{\text{isometries}\} = \{\text{lattices in } \mathbb{R}^3\},$

$\exists \Phi : V \to C^{2,\alpha}(\Sigma), \quad \Lambda \mapsto \varphi_{\Lambda}, \quad C^2$ mapping, such that

(i) $\varphi_{\Lambda_0} = 0,$

(ii) $X_{\Lambda} := X_{\varphi_{\Lambda},\Lambda}$ is a minimal surface in $(T^3, g_{\Lambda}),$

(iii) $\exists \Omega :$ a neighborhood of $X_0$ s.t. $\forall \Lambda \in V, \forall Y : \Sigma \to (T^3, g_{\Lambda})$:

minimal embedding in $\Omega$, $Y$ is congruent to $X_{\Lambda}$.

That is, in a neighborhood of $X_0$, there is a 1-1 correspondence between TPMS’s and lattices in $\mathbb{R}^3$. Hence the space of TPMS’s is (locally) 5-dimensional (up to congruence and homothety).
Theorem B (Bifurcation. K-P-S[5]). Let $U_0$ be a neighborhood of $0$ in $C^{2,\alpha}(\Sigma)$, $V_0$ be a nbd of $[\Lambda_0]$ in $T(T^3)$. Assume there is a continuous mapping $(-\varepsilon, \varepsilon) \ni s \mapsto (\varphi_s, \Lambda(s)) \in U_0 \times V_0$ s.t. $X_s := X_{\varphi_s, \Lambda(s)}$ is a minimal embedding in $(T^3, g_{\Lambda(s)})$, $(\forall s \in (-\varepsilon, \varepsilon))$.
Assume

(a) $\forall s \neq 0$, $\text{Nul}(X_s) = 3$. (i.e. there is no non-trivial nullity.)

(b) $\forall s > 0$, $\text{Ind}(X_s) - \text{Ind}(X_{-s})$ is odd. (i.e. at $s = 0$, the index jumps with an odd integer.)

Then, $s = 0$ is a bifurcation instant for the family $\{X_s\}$: i.e. in any neighborhood of $X_0$, there exists a sequence $s_n \in (-\varepsilon, \varepsilon) - \{0\}$ such that

$\exists Y_n :$ minimal embedding in $(T^3, g_{\Lambda(s_n)})$ such that

$s_n \to 0$, and $\{Y_n\} \to X_0$ in $C^{2,\alpha}$-topology, (as $n \to 0$).

$Y_n$ is not congruent to $X_{s_n}$.
3 Idea of the proofs of the main theorems

Let \( \{e_1, e_2, e_3\} \) be the canonical basis in \( \mathbb{R}^3 \), and \( \pi_\Lambda : \mathbb{R}^3 \to \mathbb{R}^3/\Lambda \) be the projection. For \( \Lambda \) and \( i = 1, 2, 3 \), set \( K_i^\Lambda = (\pi_\Lambda)_*(e_i) \). Then, \( \{K_i^\Lambda\}_i \) forms a basis of all killing vector fields in \( (T^3, g_\Lambda) \). For \( \varphi \in C^{2,\alpha}(\Sigma) \) close to 0, define a map \( f_i^{\varphi,\Lambda} : \Sigma \to \mathbb{R} \) as

\[
f_i^{\varphi,\Lambda} = g_\Lambda(K_i^\Lambda, \vec{n}_{X^{\varphi,\Lambda}}).
\]

For an embedding \( X : \Sigma \to T^3 \), denote by \( \mathcal{H}^\Lambda(X) \) the mean curvature of \( X \) in \( g_\Lambda \). For \( U_0 : \text{a nbd of } 0 \) in \( C^{2,\alpha}(\Sigma) \), \( V_0 : \text{a nbd of } [\Lambda_0] \) in \( T(T^3) \), consider a map \( \tilde{\mathcal{H}} : U_0 \times \mathbb{R}^3 \times V_0 \to C^{0,\alpha}(\Sigma) \),

\[
\tilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]) := \mathcal{H}^\Lambda(X^{\varphi,\Lambda}) + \sum_{i=1}^3 a_i f_i^{\varphi,\Lambda}.
\]

Then,

\[
\tilde{\mathcal{H}}^{-1}(0) = \{ (\varphi, 0, 0, 0, [\Lambda]) : X^{\varphi,\Lambda} \text{ is } g_\Lambda-\text{minimal} \}.
\]
Recall
\[ \tilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]) := \mathcal{H}^\Lambda(X_\varphi, \Lambda) + \sum_{i=1}^{3} a_i f_i^{\varphi, \Lambda}. \]

For \([\Lambda] \in \mathcal{T}(T^3)\), set
\[ \tilde{\mathcal{H}}_\Lambda : U_0 \times \mathbb{R}^3 \longrightarrow C^{0,\alpha}(\Sigma), \quad \tilde{\mathcal{H}}_\Lambda(\varphi, a_1, a_2, a_3) := \tilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]). \]

Assume \(\tilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0) = 0\). Consider
\[ T_{\varphi, \Lambda} := d\tilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0) : C^{2,\alpha}(\Sigma) \times \mathbb{R}^3 \longrightarrow C^{0,\alpha}(\Sigma). \]

Then, \(\forall (\psi, b_1, b_2, b_3) \in C^{2,\alpha}(\Sigma) \times \mathbb{R}^3\),
\[ T_{\varphi, \Lambda}(\psi, b_1, b_2, b_3) = J_{x_{\varphi, \Lambda}}(\psi) + \sum_{i=1}^{3} b_i f_i^{\varphi, \Lambda}, \]
where \(J_{x_{\varphi, \Lambda}}\) is the Jacobi operator of \(X_{\varphi, \Lambda}\). \(T_{\varphi, \Lambda}\) is Fredholm with index 3.

\(T_{\varphi, \Lambda}\) is surjective. \(\iff\) \(X_{\varphi, \Lambda}\) is \(g_\Lambda\)-minimal with nullity 3.

We apply the bifurcation theory (e.g. Kato[3], [4]) to \(\tilde{\mathcal{H}}_\Lambda\). \qed
4 Applications to explicit examples

(Most of pictures below were drawn by Prof. Shoichi Fujimori.)

Examples of 1-parameter families of TPMS’s:

[H-family]  [tCLP-family]  [tD-family]

[rPD-family (Karcher’s TT surfaces)]  [tP-family]
【Representation of rPD-family】(Use Weierstrass formula.)

\[ M_a := \left\{ (w, \zeta) \in \mathbb{C}^2 \mid \zeta^2 = w(w^3 - a^3)(w^3 + a^{-3}) \right\}, \ a > 0 : \text{Riemann surface.} \]

\[ X_a(w) := \text{Re} \int_{w_0}^{w} \left( 1 - w^2, i(1 + w^2), 2w \right) \zeta^{-1} dw, \ w \in M_a. \]

\[ \begin{align*}
  a &= 1/\sqrt{2}, \ b = 14: \text{P} \\
  a &= \sqrt{2}, \ b = 14: \text{D}
\end{align*} \]

【Representations of tP-family and tD-family】

\[ N_b := \left\{ (w, \zeta) \in \mathbb{C}^2 \mid \zeta^2 = w^8 + bw^4 + 1 \right\}, \ b \in (2, +\infty): \text{Riemann surface.} \]

For \( w \in N_b, \)

\[ \begin{align*}
  \text{tP-family: } \varphi_b(w) &= \text{Re} \int_{w_0}^{w} \left( 1 - w^2, i(1 + w^2), 2w \right) \zeta^{-1} dw, \\
  \text{tD-family: } \psi_b(w) &= \text{Re} \int_{w_0}^{w} i(1 - w^2, i(1 + w^2), 2w) \zeta^{-1} dw.
\end{align*} \]
We can apply our main theorems to explicit examples. There is a method to compute the nullities and the indices of TPMS’s given by Ejiri-Shoda[2], which includes computation of eigenvalues of $18 \times 18$ matrices whose elements are elliptic integrals! So we need help of numerical computation.

Also, we can find eigenfunctions belonging to zero eigenvalue by using a method given by Montiel-Ros (1991[7]), Ejiri-Kotani (1993).

Example 4.1 (Application with numerical computation) There is one bifurcation instant for the H-family, and two bifurcation instants for the rPD-family, the tP-family, and the tD-family.

This means that there is possibility that we found the existence of new TPMS’s which are close to known examples.
Schwarz P-surface with Lines

The P-surface can be constructed by solving the Plateau problem for a 4-gon with corners at the vertices of a regular octahedron. The resulting surface is then extended by 180º rotations about the straight boundary lines.

See also:
- D-surface
- H-surface
- CLP-surface
- P-surface with handle
- Mathematica Notebook

Scherk's surface

new'surface'frpm'rPD

homothetic to original surfaces

new surfaces

jump of index

new surface frpm rPD
5 Future subjects

(1) Try to verify the application results obtained by using numerical computations.

(2) Find explicit representations of the new surfaces.

(3) Study the geometry of the surfaces in the bifurcating branches: eg. symmetry-breaking property.
   
   Ex. Bifurcation from the rPD-family: Variation vector field should be the zero eigenfunction.

(4) Study the stability/instability of minimal surfaces in the bifurcating branches.
References


