Stability for the Yamabe equation on non-compact manifolds

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November, 2015
$M$ closed smooth manifold

$g = \langle \ , \ \rangle_x$ \hspace{1cm} Riemannian metric on $M$.

Look for metrics of constant scalar curvature in the conformal class

$$[g] = \{ f . g : f : M \to \mathbb{R}_{>0} \}.$$ 

It amounts to solving the Yamabe equation:

$$-a_n \Delta_g u + s_g u = \lambda u^{p-1}$$

$$a_n = \frac{4(n-1)}{n-2}, \ s_g \text{ the scalar curvature}, \ p = p_n = \frac{2n}{n-2},$$

$$\lambda \in \mathbb{R} \text{ is the scalar curvature of } u^{p-2}g.$$
Sign of $\lambda$ is determined by $[g]$. Yamabe equation is the Euler-Lagrange equation for the Hilbert-Einstein functional restricted to $[g]$: 

$$S(h) = \frac{\int_M s_h \, dvol_h}{Vol(M, h)^{n-2}}$$

Let the Yamabe constant of $(M, [g])$ be

$$Y(M, [g]) = \inf_{h \in [g]} S(h) = \inf_f \frac{\int_M a_n \| \nabla f \|^2 + s_g f^2 \, dvol_g}{(\int_M f^p \, dvol_g)^{2/p}}$$
1. Infimum in the definition of $Y(M, [g])$ is always achieved H. Yamabe-N. Trudinger-T. Aubin-R. Schoen. There is always at least one (volume 1) solution of the Yamabe equation.

2. Solution is unique if $Y(M, [g]) \leq 0$.

3. Solution is unique if $g$ is Einstein (M. Obata).

4. In general multiple solutions when $Y(M,[g]) > 0$.
(M, g) non-compact Riemannian manifold
Consider the case $s_g$ positive, constant.
Define its Yamabe constant by:

\[
Y(M, [g]) = \inf_f Y_g(f) = \inf_f \frac{\int_M a_n \|\nabla f\|^2 + s_g f^2 dv_{\text{vol} g}}{(\int_M f^p dv_{\text{vol} g})^{2/p}}
\]

\[
= \inf_f \frac{Q(f)}{\|f\|_p^2}.
\]

\(f \neq 0, f \in L^2_1(M, g)\), assume that the embedding \(L^2_1 \subset L^p\) holds (positive injectivity radius and bounded Ricci curvature).
Questions

1. Compute (O. Kobayashi, R. Schoen)

\[ Y(M) = \sup \{ [g] \} \, Y(M, [g]) \leq Y(S^n, [g^n]) \]

(T. Aubin).

2. Find all solutions of the Yamabe equation on \([g]\).

Can we solved the Yamabe equation on \((S^n \times S^m, g^n_0 + Tg^m_0)\) \((T > 0, n, m \geq 2)\), compute the Yamabe constants?

Solution is not unique for \(T\) big (or small).
Noncompact manifolds appear:

$$\lim_{T \to \infty} Y(S^n \times S^m, [g^n_0 + Tg^m_0]) = Y(S^n \times \mathbb{R}^m, g^n_0 + g_E)$$

(K. Akutagawa-L. Florit-J. Petean)

It is fundamental for understanding the behavior of $Y(M)$ under codim $k \geq 3$-surgery

$$Y(\overline{M}) \geq \inf \{ Y(M), c(n, k) \}$$

(B. Ammann-M. Dahl-E. Humbert), generalizing the case of 0-surgery (Kobayashi) $Y(\overline{M}) \geq Y(M)$. 

Stability for the Yamabe equation on non-compact manifolds
(\(M, g\)) with constant positive scalar curvature, \(u\) smooth function on \(M\)

\[
h_u(t) = Y_g(1 + tu)
\]

Since \(s_g\) is constant \(h_u'(0) = 0\). If \(g\) were a Yamabe metric then \(h_u''(0) \geq 0\).

In general if the inequality holds for all \(u\) we say that \(g\) is a stable solution of the Yamabe equation.

Standard computation:

\[
h_u''(0) = \frac{2}{V^{2/p}} \left( Q(u) - s_g(p - 1) \int_M u^2 + \frac{(p - 2)s_g}{V} \left( \int_M u \right)^2 \right)
\]
Then $g$ is stable if for all $u$ such that $\int u = 0$ one has
\[ a_n \|\nabla u\|_2^2 - (p - 2) s_g \|u\|_2^2 \geq 0 \]
which means
\[ \lambda_1(g) \geq \frac{s_g}{n - 1}. \]
In particular this holds for any Yamabe metric.
For some canonical metric $g$ we might want to study stability for other solutions of the Yamabe equation, and write everything in terms of $g$.
Consider for instance $(S^3 \times S^1, g^3_0 + Tdt^2)$ (all solutions known (O. Kobayashi, R. Schoen), minimizer is the only stable one) $(S^2 \times S^2, g^2_0 + Tg^2_0)$ (there are solutions computed numerically, candidate for minimizer).
Let \((X, h)\) has constant positive scalar curvature, \(f \in L^2_1(X)\) be a smooth positive solution of the Yamabe equation. Then \(f\) is stable if for all \(u \in L^2_1(X)\) such that \(\int f^{p-1} u \, dv_h = 0\) we have

\[
\frac{Q_h(u)}{\int_X f^{p-2} u^2 \, dv_h} \geq (p - 1) \frac{Q_h(f)}{\|f\|_p^p}
\]

Let

\[
\alpha(X, h, f) = \inf_{u \in N(h, f)} \frac{Q_h(u)}{\int_X f^{p-2} u^2 \, dv_h}
\]
Consider \((M^m, g)\) closed with positive constant scalar curvature and \((X, h) = (M \times \mathbb{R}^n, g + g^n_E)\).

\(f\) a critical point of \(Y_h\) which is a smooth radial decreasing positive function on \(\mathbb{R}^n\).

**Theorem**

*Exists* \(u \in N(g + g^n_E, f)\) *which achieves the infimum in the definition of* \(\alpha(M \times \mathbb{R}^n, g + g^n_E, f)\).* *Every minimizer is a smooth function which solves the equation*

\[
-a_n \Delta u + (s_g - \alpha f^{p-2})u = 0
\]

(1)

*The space of solutions of this equation is finite dimensional.*
(\(M, g\)) closed, constant positive scalar curvature.  \(g^n_E\) the Euclidean metric on \(\mathbb{R}^n\). Assume that \(m, n \geq 2\).

For a Riemannian product \((Z, G) = (M_1 \times M_2, g + h)\) consider the restriction of \(Y_G\) to functions on one of the variables and let

\[
Y_{M_i}(Z, G) = \inf_{u \in L^2_1(M_i)} Y_G(u).
\]

\(Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g^n_E)\) can be computed in terms of the best constant in the Gagliardo-Nirenberg inequality (K. Akutagawa, L. Florit, J. Petean):

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Gagliardo-Nirenberg inequality

\[ \|u\|_{p_{m+n}}^2 \leq \sigma \|\nabla u\|_{2}^{\frac{2n}{m+n}} \|u\|_{2}^{\frac{2m}{m+n}}. \]

invariant by replacing \( u \) by \( cu_\lambda(x) = cu(\lambda x) \). The best constant is

\[
\sigma_{m,n} = \left( \inf_{u \in C_0^\infty(\mathbb{R}^n) - \{0\}} \frac{\|\nabla u\|_{2}^{\frac{2n}{m+n}} \|u\|_{2}^{\frac{2m}{m+n}}}{\|u\|_{p_{m+n}}^2} \right)^{-1}
\]
The infimum is actually achieved. The minimizer is a solution of the Euler-Lagrange equation of the functional in parenthesis:

$$-n\Delta u + m\frac{\|\nabla u\|^2}{\|u\|^2} u - (m + n)\frac{\|\nabla u\|^2}{\|u\|^p} u^{p-1} = 0. \quad (2)$$

By invariance if a function $u$ is a minimizer so is $cu_\lambda$ given by $cu_\lambda(x) = cu(\lambda x)$ for any constants $c, \lambda \in \mathbb{R}_{>0}$. By picking $c, \lambda$ appropriately we can choose the (constant) coefficients appearing in the equation. In particular one would have a solution of

$$-\Delta u + u - u^{p-1} = 0 \quad (3)$$
It is known since classical work of Gidas-Ni-Nirenberg that all solutions of (3), which are positive and vanish at infinity, are radial functions. It is also known the existence of a radial solution. Moreover, M. K. Kwong proved that such a solution is unique.

In our situation we will prefer to first choose $\lambda$ so that
\[
 am + n \|\nabla u\|_2^2 = n s g \|u\|_2^2
\]
and then pick $c$ so that
\[
 (m + n) a_{m+n} \|\nabla u\|_2^2 = s g n \|u\|_p^p.
\]
Then the resulting function $f_K$ satisfies
\[
 -a_{m+n} \Delta f_K + s g f_K = s g f_K^{p-1} \tag{4}
\]
The metric $g_K = f_K^{p-2}(g + g_E^n)$ has scalar curvature $s_{g_K} = s_g$. A minimizer for $Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n)$ must be a solution of (4). And by the previous comments the solution is unique, so actually the solution $f_{m,n,s,g}^K$ is the unique minimizer for $Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n)$. We have

$$Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n) = s_g \text{Vol}(g_K)^{\frac{2}{m+n}}.$$ 

which can be expressed in terms of $\sigma_{m,n}$ and the volume of $g$. Stability for the Yamabe equation on non-compact manifolds
Let \( g \) be a Riemannian metric on the closed \( m \)-manifold \( M \) of constant scalar curvature \( s_g = m(m - 1) \). To simplify we will use the notation \( G = g + g^p_E \), \( N = m + n \). Let \( f : \mathbb{R}^n \to \mathbb{R}_{>0} \) be the unique solution of equation (4) discussed in the previous section.

Note that \( Q_G(f) = m(m - 1) \| f \|_p^p \ V \).

**Lemma**

If \( \alpha = \alpha(M \times \mathbb{R}^n, G, f) < (p - 1)m(m - 1) \) then it is realized by a function \( u(y, x) = a(y)b(x) \) where \( a : M \to \mathbb{R} \), \( -\Delta_g a = \lambda_1 a \), \( \lambda_1 \) is the first positive eigenvalue, and \( b \in L^2_1(\mathbb{R}^n) \) satisfies the equation:

\[
-a_N \Delta b + \left( -a_N \lambda_1 + m(m - 1) - \alpha f^{p-2} \right) b = 0 \quad (5)
\]
By previous Theorem there exists a minimizer and it is a solution of the equation

\[-a_N \Delta u + \left( m(m-1) - \alpha f^{p-2} \right) u = 0\]

(and the space of solutions is finite dimensional). \(f\) depends only on \(\mathbb{R}^n\): if \(u\) is a solution then \(\Delta_g u\) is also a solution. It follows that there is a finite number of linearly independent \(\Delta_g\)-eigenfunctions \(a_1(y), \ldots, a_k(y)\), \(\Delta_g a_i = \lambda_i a_i\ (\lambda_i \leq 0)\), such that \(u = \sum a_i(y)b_i(x)\) for some \(b_i : \mathbb{R}^n \to \mathbb{R}\). Then

\[\sum_{i=1}^k -a_N(\lambda_i a_i(y)b_i(x)) + a_i(y)\Delta b_i(x)) + (m(m-1) - \alpha f^{p-2}) a_i(y)b_i(x) = 0.\]
But then since the $a_i$ are linearly independent it follows that for each $i$

$$-a_N(\lambda_i b_i(x) + \Delta b_i(x)) + \left( m(m-1) - \alpha f^{p-2} \right) b_i(x) = 0.$$  

So $a_i b_i$ is also a solution for each $i$: there is a minimizer of the form $a(y)b(x)$ with $-\Delta_g a = \lambda a$ for some $\lambda \geq 0$. 

If $\lambda = 0$ we take $a = 1$ and then we must have $\int_{\mathbb{R}^n} b f^{p-1} \, dx = 0$. 

Since $f$ is a $\text{Y}_{\mathbb{R}^n}$-minimizer it is $\mathbb{R}^n$-stable we would have

$$\alpha(M \times \mathbb{R}^n, G, f) \geq (p-1) \frac{E_G(f)}{\|f\|^p_p} = (p-1) m(m-1)$$

If $\lambda > 0$ note that

$$\frac{Q_G(ab)}{\int_{\mathbb{R}^n} f^{p-2} a^2 b^2} = \frac{\int_{\mathbb{R}^n} a_N \|\nabla b\|^2_2 + s_g b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2} + a_N \lambda \frac{\int_{\mathbb{R}^n} b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2}.$$
Therefore $f$ is unstable if and only if exists $b \in L^2_1(\mathbb{R}^n) - \{0\}$

$$\int_{\mathbb{R}^n} a_N \| \nabla b \|^2_2 + m(m - 1)b^2 \int_{\mathbb{R}^n} f^{p-2}b^2 + a_N \lambda_1 \int_{\mathbb{R}^n} b^2 \int_{\mathbb{R}^n} f^{p-2}b^2 < (p-1)m(m-1)$$

**Lemma**

For each $\lambda \geq 0$

$$A(\lambda) = \inf_{b \in L^2_1(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} a_N \| \nabla b \|^2_2 + sgb^2 \int_{\mathbb{R}^n} f^{p-2}b^2}{\int_{\mathbb{R}^n} f^{p-2}b^2} + \lambda \frac{\int_{\mathbb{R}^n} b^2 \int_{\mathbb{R}^n} f^{p-2}b^2}{\int_{\mathbb{R}^n} f^{p-2}b^2}$$

is realized by a radial decreasing function.
Then $A(\lambda)$ is strictly increasing function of $\lambda$. 
$A(0) \leq m(m - 1)$ (take $b = f$) and $A(\infty) = \infty$. 
Then exists a unique $\lambda = \lambda_{m,n}$ such that 
$A(\lambda_{m,n}) = (p - 1)m(m - 1)$. 
It follows that $f$ is unstable if and only if $\lambda_1 < \lambda_{m,n}$. 
Recall that if $g$ is a Yamabe metric on $M^m$ with scalar curvature $m(m - 1)$ then $\lambda_1(g) \geq m$.

**Theorem**

*If $\lambda_{m,n} \leq m$ then $f^{p-2}(g + g^n_{\mathcal{E}})$ is stable for any Yamabe metric $g$*
$\lambda_{m,n}$ can be computed numerically. The minimizer $b$ for $A(\lambda_{m,n})$ is a solution of

$$-a_N \Delta b + (m(m-1) + a_N \lambda_{m,n}) b = (p - 1)m(m-1)f^{p-2}b.$$  

In general consider the equation

$$-\Delta b + Kb = Cf^{p-2}b,$$  \hspace{1cm} (6)

where $C = (p - 1)m(m - 1)/a_N$ and $K$ is a (variable) positive constant. A radial solution is given by a solution of the ordinary linear differential equation:

$$u''(t) + \frac{n-1}{t} u'(t) + (Cf^{p-2} - K)u(t) = 0$$  \hspace{1cm} (7)

with $u(0) = 1$, $u'(0) = 0$.  

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Denote the solution $u$ by $u_K$. We have 3 possibilities:

a) $u_K$ is always decreasing and positive.

b) $u_K(t) = 0$ for some $t > 0$.

c) $u_K$ has a local minimum at some $t \geq t_0$.

By Sturm comparison, $K_1 < K_2$, if the solution $u_{K_1}$ verifies (c) then the solution $u_{K_2}$ also verifies (c). If $u_{K_2}$ verifies (b) then $u_{K_1}$ also verifies (b). Moreover if $u_{K_2}$ verifies (a) then $u_{K_1}$ verifies (b).
It follows that for $\lambda = \lambda_{m,n}$ the equation

$$u''(t) + \frac{n-1}{t} u'(t) + \left( Cf^{p-2} - \left( \frac{m(m-1)}{a_N} + \lambda \right) \right) u(t) = 0 \quad (8)$$

is positive and decreasing. For $\lambda > \lambda_{m,n}$ the solution has a local minimum and for $\lambda < \lambda_{m,n}$ has a 0 at finite time. The function $f$ can be computed numerically and then for a fixed $\lambda$ one can compute numerically the solution of (8) and check whether $\lambda < \lambda_{m,n}$ or $\lambda > \lambda_{m,n}$.
For example:

\[ \lambda_{2,2} \approx 1.8041 \]
\[ \lambda_{3,2} \approx 2.9183 \]
\[ \lambda_{4,2} \approx 3.9553 \]
\[ \lambda_{5,2} \approx 4.9718 \]
Thank you !!