Stability for the Yamabe equation on non-compact manifolds

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Yamabe equation: Introduction

M closed smooth manifold

$g = \langle\ ,\ \rangle_x$ Riemannian metric on M.

Look for metrics of constant scalar curvature in the conformal class

$$[g] = \{f.g : f : M \to \mathbb{R}_{>0}\}.$$

It amounts to solving the Yamabe equation:

$$-a_n \Delta_g u + s_g u = \lambda\ u^{p-1}$$

$a_n = \frac{4(n-1)}{n-2}$, s_g the scalar curvature, $p = p_n = \frac{2n}{n-2}$,

$\lambda \in \mathbb{R}$ is the scalar curvature of $u^{p-2} g$.

Sign of λ is determined by $[g]$.

Yamabe equation is the Euler-Lagrange equation for the Hilbert-Einstein functional restricted to $[g]$:

$$S(h) = \frac{\int_M s_h \, dvol_h}{Vol(M,h)^{\frac{n-2}{n}}}$$

Let the Yamabe constant of $(M, [g])$ be

$$Y(M,[g]) = \inf_{h \in [g]} S(h) = \inf_f \frac{\int_M a_n \|\nabla f\|^2 + s_g f^2 dvol_g}{\left(\int_M f^p dvol_g\right)^{2/p}}$$

1. Infimum in the definition of $Y(M,[g])$ is always achived H. Yamabe-N. Trudinger-T. Aubin-R. Schoen. There is always at least one (volume 1) solution of the Yamabe equation.
2. Solution is unique if $Y(M,[g]) \leq 0$.
3. Solution is unique if g is Einstein (M. Obata).
4. In general multiple solutions when Y(M,[g]) >0

(M,g) non-compact Riemannian manifold

Consider the case s_g positive, constant.

Define its Yamabe constant by:

$$Y(M,[g]) = \inf_f Y_g(f) = \inf_f \frac{\int_M a_n \|\nabla f\|^2 + s_g f^2 dvol_g}{\left(\int_M f^p dvol_g\right)^{2/p}}$$

$$= \inf_f \frac{Q(f)}{\|f\|_p^2}.$$

$f \neq 0$, $f \in L_1^2(M,g)$, assume that the embedding $L_1^2 \subset L^p$ holds (positive injectivity radius and bounded Ricci curvature).

Questions

1. Compute (O. Kobayashi, R. Schoen)

   $Y(M) = \sup_{\{[g]\}} Y(M,[g]) \le Y(S^n,[g_0^n])$

   (T. Aubin).

2. Find all solutions of the Yamabe equation on $[g]$.

Can we solved the Yamabe equation on $(S^n \times S^m, g_0^n + T g_0^m)$ ($T > 0$, $n, m \ge 2$), compute the Yamabe constants ?

Solution is not unique for T big (or small).

Noncompact manifolds appear:

$$\lim_{T\to\infty} Y(S^n \times S^m, [g_0^n + Tg_0^m])) = Y(S^n \times \mathbb{R}^m, g_0^n + g_E)$$

(K. Akutagawa-L. Florit-J. Petean)

It is fundamental for understanding the behavior of $Y(M)$ under codim $k \geq 3$-surgery

$$Y(\overline{M}) \geq \inf\{Y(M), c(n,k)\}$$

(B. Ammann-M. Dahl-E. Humbert), generalizing the case of 0-surgery (Kobayashi) $Y(\overline{M}) \geq Y(M)$.

(M, g) with constant positive scalar curvature, u smooth function on M

$$h_u(t) = Y_g(1 + tu)$$

Since s_g is constant $h'_u(0) = 0$. If g were a Yamabe metric then $h''_u(0) \geq 0$.
In general if the inequality holds for all u we say that g is a stable solution of the Yamabe equation.
Standard computation:

$$h''_u(0) = \frac{2}{V^{2/p}} \left(Q(u) - s_g(p-1) \int_M u^2 + \frac{(p-2)s_g}{V} \left(\int_M u \right)^2 \right)$$

Then g is stable if for all u such that $\int u = 0$ one has $a_n \|\nabla u\|_2^2 - (p-2)s_g\|u\|_2^2 \geq 0$ which means

$$\lambda_1(g) \geq \frac{s_g}{n-1}.$$

In particular this holds for any Yamabe metric.

For some canonical metric g we might want to study stability for other solutions of the Yamabe equation, and write everything in terms of g.

Consider for instance $(S^3 \times S^1, g_0^3 + Tdt^2)$ (all solutions known (O. Kobayashi, R. Schoen), minimizer is the only stable one)

$(S^2 \times S^2, g_0^2 + Tg_0^2)$ (there are solutions computed numerically, candidate for minimizer).

Let (X, h) has constant positive scalar curvature, $f \in L_1^2(X)$ be a smooth positive solution of the Yamabe equation.
Then f is stable if for all $u \in L_1^2(X)$ such that $\int f^{p-1} u \, dv_h = 0$ we have

$$\frac{Q_h(u)}{\int_X f^{p-2} u^2 dv_h} \geq (p-1) \frac{Q_h(f)}{\|f\|_p^p}$$

Let

$$\alpha(X, h, f) = \inf_{u \in N(h,f)} \frac{Q_h(u)}{\int_X f^{p-2} u^2 dv_h}.$$

Consider (M^m, g) closed with positive constant scalar curvature and $(X, h) = (M \times \mathbb{R}^n, g + g_E^n)$.
f a critical point of Y_h which is a smooth radial decreasing positive function on $\mathbb{R}^n$.

Theorem

Exists $u \in N(g + g_E^n, f)$ which achieves the infimum in the definition of $\alpha(M \times \mathbb{R}^n, g + g_E^n, f)$. Every minimizer is a smooth function which solves the equation

$$-a_n \Delta u + (s_g - \alpha f^{p-2})u = 0 \qquad (1)$$

The space of solutions of this equation is finite dimensional.

Stability, $Y_{\mathbb{R}^n}$-minimizer

(M, g) closed, constant positive scalar curvature. g_E^n the Euclidean metric on $\mathbb{R}^n$. Assume that $m, n \geq 2$.

For a Riemannian product $(Z, G) = (M_1 \times M_2, g + h)$ consider the restricion of Y_G to functions on one of the variables and let

$$Y_{M_i}(Z, G) = \inf_{u \in L_1^2(M_i)} Y_G(u).$$

$Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n)$ can be computed in terms of the best constant in the Gagliardo-Nirenberg inequality (K. Akutagawa, L. Florit, J. Petean):

Gagliardo-Nirenberg inequality

$$\|u\|_{p_{m+n}}^2 \leq \sigma \|\nabla u\|_2^{\frac{2n}{m+n}} \|u\|_2^{\frac{2m}{m+n}}.$$

invariant by replacing u by $cu_\lambda(x) = cu(\lambda x)$. The best constant is

$$\sigma_{m,n} = \left(\inf_{u \in C_0^\infty(\mathbb{R}^n) - \{0\}} \frac{\|\nabla u\|_2^{\frac{2n}{m+n}} \|u\|_2^{\frac{2m}{m+n}}}{\|u\|_{p_{m+n}}^2} \right)^{-1}.$$

The infimum is actually achieved. The minimizer is a solution of the Euler-Lagrange equation of the functional in parenthesis:

$$-n\Delta u + m\frac{\|\nabla u\|_2^2}{\|u\|_2^2}u - (m+n)\frac{\|\nabla u\|_2^2}{\|u\|_p^p}u^{p-1} = 0. \qquad (2)$$

By invariance if a function u is a minimizer so is cu_λ given by $cu_\lambda(x) = cu(\lambda x)$ for any constants $c, \lambda \in \mathbb{R}_{>0}$.
By picking c, λ appriopriately we can choose the (constant) coefficients appearing in the equation. In particular one would have a solution of

$$-\Delta u + u - u^{p-1} = 0 \qquad (3)$$

It is known since classical work of Gidas-Ni-Nirenberg that all solutions of (3), which are positive and vanish at infinity, are radial functions. It is also known the existence of a radial solution. Moreover, M. K. Kwong proved that such a solution is unique.
In our situation we will prefer to first choose λ so that $a_{m+n} m \|\nabla u\|_2^2 = n s_g \|u\|_2^2$ and then pick c so that $(m+n) a_{m+n} \|\nabla u\|_2^2 = s_g n \|u\|_p^p$. Then the resulting function f_K satisfies

$$-a_{m+n} \Delta f_K + s_g \ f_K = s_g \ f_K^{p-1} \qquad (4)$$

The metric $g_K = f_K^{p-2}(g + g_E^n)$ has scalar curvature $s_{g_K} = s_g$ A minimizer for $Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n)$ must be a solution of (4). And by the previous comments the solution is unique, so actually the solution f_K^{m,n,s_g} is the unique minimizer for $Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n)$. We have

$$Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_E^n) = s_g Vol(g_K)^{\frac{2}{m+n}}.$$

which can be expressed in terms of $\sigma_{m,n}$ and the volume of g.

Let g be a Riemannian metric on the closed m-manifold M of constant scalar curvature $s_g = m(m-1)$. To simplify we will use the notation $G = g + g_E^n$, $N = m + n$ Let $f : \mathbb{R}^n \to \mathbb{R}_{>0}$ be the unique solution of equation (4) discussed in the previous section.
Note that $Q_G(f) = m(m-1)\|f\|_p^p \ V$.

Lemma

If $\alpha = \alpha(M \times \mathbb{R}^n, G, f) < (p-1)m(m-1)$ then it is realized by a function $u(y,x) = a(y)b(x)$ where $a : M \to \mathbb{R}$, $-\Delta_g a = \lambda_1 a$, λ_1 is the first positive eigenvalue, and $b \in L_1^2(\mathbb{R}^n)$ satisties the equation:

$$-a_N \Delta b + \left(-a_N \lambda_1 + m(m-1) - \alpha f^{p-2}\right) b = 0 \qquad (5)$$

By previous Theorem there exists a minimizer and it is a solution of the equation

$$-a_N \Delta u + \left(m(m-1) - \alpha f^{p-2} \right) u = 0$$

(and the space of solutions is finite dimensional). f depends only on $\mathbb{R}^n$: if u is a solution then $\Delta_g u$ is also a solution.
It follows that there is a finite number of linearly independent Δ_g- eigenfunctions $a_1(y), ..., a_k(y)$, $\Delta_g a_i = \lambda_i a_i$ ($\lambda_i \leq 0$), such that $u = \Sigma a_i(y) b_i(x)$ for some $b_i : \mathbb{R}^n \to \mathbb{R}$.
Then

$$\Sigma_{i=1}^{k} - a_N(\lambda_i a_i(y) b_i(x)$$

$$+ a_i(y) \Delta b_i(x)) + \ (m(m-1) - \alpha f^{p-2})\ a_i(y) b_i(x) = 0.$$

But then since the a_i are linearly independent it follows that for each i

$$-a_N(\lambda_i b_i(x) + \Delta b_i(x)) + \left(m(m-1) - \alpha f^{p-2}\right) b_i(x) = 0.$$

So $a_i b_i$ is also a solution for each i: there is a minimizer of the form $a(y)b(x)$ with $-\Delta_g a = \lambda a$ for some $\lambda \geq 0$.
If $\lambda = 0$ we take $a = 1$ and then we must have $\int_{\mathbb{R}^n} b f^{p-1} dx = 0$.
Since f is a $Y_{\mathbb{R}^n}$-minimizer it is $\mathbb{R}^n$-stable we would have

$$\alpha(M \times \mathbb{R}^n, G, f) \geq (p-1)\frac{E_G(f)}{\|f\|_p^p} = (p-1)m(m-1)$$

If $\lambda > 0$ note that

$$\frac{Q_G(ab)}{\int_{\mathbb{R}^n} f^{p-2} a^2 b^2} = \frac{\int_{\mathbb{R}^n} a_N \|\nabla b\|_2^2 + s_g b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2} + a_N \lambda \frac{\int_{\mathbb{R}^n} b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2}.$$

Therefore f is unstable if and only if exists $b \in L_1^2(\mathbb{R}^n) - \{0\}$

$$\frac{\int_{\mathbb{R}^n} a_N \|\nabla b\|_2^2 + m(m-1)b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2} + a_N \lambda_1 \frac{\int_{\mathbb{R}^n} b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2} < (p-1)m(m-1)$$

Lemma

For each $\lambda \geq 0$

$$A(\lambda) = \inf_{b \in L_1^2(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} a_N \|\nabla b\|_2^2 + s_g b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2} + \lambda \frac{\int_{\mathbb{R}^n} b^2}{\int_{\mathbb{R}^n} f^{p-2} b^2}$$

is realized by a radial decreasing function.

Then $A(\lambda)$ is strictly increasing function of λ.
$A(0) \leq m(m-1)$ (take $b = f$) and $A(\infty) = \infty$.
Then exists a unique $\lambda = \lambda_{m,n}$ such that
$A(\lambda_{m,n}) = (p-1)m(m-1)$.
It follows that f is unstable if and only if $\lambda_1 < \lambda_{m,n}$.
Recall that if g is a Yamabe metric on $M^m$with scalar curvature $m(m-1)$ then $\lambda_1(g) \geq m$.

Theorem

If $\lambda_{m,n} \leq m$ then $f^{p-2}(g + g_E^n)$ is stable for any Yamabe metric g

$\lambda_{m,n}$ can be computed numerically. The minimizer b for $A(\lambda_{m,n})$ is a solution of

$$-a_N \Delta b + (m(m-1) + a_N \lambda_{m,n}) b = (p-1) m(m-1) f^{p-2} b.$$

In general consider the equation

$$-\Delta b + K b = C f^{p-2} b, \tag{6}$$

where $C = (p-1)m(m-1)/a_N$ and K is a (variable) positive constant. A radial solution is given by a solution of the ordinary linear differential equation:

$$u''(t) + \frac{n-1}{t} u'(t) + (C f^{p-2} - K) u(t) = 0 \tag{7}$$

with $u(0) = 1$, $u'(0) = 0$.

Denote the solution u by u_K. We have 3 possibilities:

a) u_K is always decreasing and positive.

b) $u_K(t) = 0$ for some $t > 0$.

c) u_K has a local minimum at some $t \geq t_0$.

By Sturm comparison, $K_1 < K_2$, if the solution u_{K_1} verifies (c) then the solution u_{K_2} also verifies (c). If u_{K_2} verifies (b) then u_{K_1} also verifies (b). Moreover if u_{K_2} verifies (a) then u_{K_1} verifies (b).

It follows that for $\lambda = \lambda_{m,n}$ the equation

$$u''(t) + \frac{n-1}{t} u'(t) + \left(Cf^{p-2} - \left(\frac{m(m-1)}{a_N} + \lambda \right) \right) u(t) = 0 \quad (8)$$

is positive and decreasing.
For $\lambda > \lambda_{m,n}$ the solution has a local minimum and for $\lambda < \lambda_{m,n}$ has a 0 at finite time. The function f can be computed numerically and then for a fixed λ one can compute numerically the solution of (8) and check whether $\lambda < \lambda_{m,n}$ or $\lambda > \lambda_{m,n}$.

For example:

$$\lambda_{2,2} \approx 1.8041$$

$$\lambda_{3,2} \approx 2.9183$$

$$\lambda_{4,2} \approx 3.9553$$

$$\lambda_{5,2} \approx 4.9718$$

Thank you !!