概複素多様体上の放物型フローについて
On some parabolic flows on almost complex manifolds

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Abstract

We introduce some parabolic flows on almost complex manifolds. First, we define an almost Hermitian flow which starts at an almost Hermitian metric. The motivation is to extend the identifiabilty theorem for a pluriclosed flow, which was proven by Streets and Tian. In this regard, we define another parabolic flow which is called an almost Hermitian curvature flow and prove that an almost Hermitian flow and an almost Hermitian curvature flow are equivalent. Second, we define an almost pluriclosed flow, which starts at an almost pluriclosed metric and preserves the almost pluriclosedness. The difference between an almost pluriclosed flow and an almost Hermitian flow is that for the first one, we use the almost pluriclosedness in the proof of the short-time existence of the flow, on the other hand, we can prove the short-time existence without using the almost pluriclosedness for the second one. Third, we introduce a scalar Calabi-type flow in almost Hermitian geometry. This is a parabolic flow of almost Hermitian metrics which evolves an initial metric along the second derivative of the Chern scalar curvature. We show that the flow has a unique short-time solution and also show a stability result when the background metric is quasi-Kähler with constant scalar curvature. Finally, we introduce a parabolic flow of almost balanced metrics. We show that the flow has a unique solution on compact almost Hermitian manifolds and that if the initial structure is Kähler, then the flow reduces to the Calabi flow.

Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in whole this article.

1 An almost Hermitian flow and an almost Hermitian curvature flow

In [12] and [14], Streets and Tian introduced a parabolic evolution equation of pluriclosed metrics on a compact Hermitian manifold, which is called the pluriclosed flow. At first, they asked whether or not it is possible to prove classification results in higher dimensions for complex non-Kähler manifolds using geometric evolution equations as in the case that the Ricci flow was used for proving uniformization of Riemann surfaces. They tried to have a parabolic flow such that it preserves Hermitianness and as much additional structure as possible and also is as close to the Kähler-Ricci flow as possible. Since a pluriclosed form ω is locally given by $\omega = \partial\eta + \bar{\partial}\bar{\eta}$ for some $\eta \in \Lambda^{0,1}$ (cf. [11, Lemma 3.9]), they concluded that it is natural to define a flow of pluriclosed metrics using a second order

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closed $(1,1)$-form (the Chern curvature form) and a first-order $(0,1)$-form. From this point of view, they defined the pluriclosed flow starting at a pluriclosed metric. They also introduced the Hermitian curvature flow with a quadratic Q^1 in the torsion of the Chern connection in [12]: Let (M, J) be a compact complex manifold with pluriclosed metric g_0. They have proven that the following evolution equation has a unique solution $g(t)$ on M (cf. [12, Theorem 1.2]).

$$(\mathrm{HCF})_{Q^1} \qquad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) + Q^1(g(t)), \\ \\ g(0) = g_0. \end{cases}$$

Streets and Tian showed that the solution of the pluriclosed flow coincides with the solution of $(\mathrm{HCF})_{Q^1}$ as follows (cf. [12, Proposition 3.3]):

Proposition 1.1. (Streets-Tian identifiability theorem) Let (M, J) be a compact complex manifold with pluriclosed metric $\tilde{g}$. The solution to $(\mathrm{HCF})_{Q^1}$ with pluriclosed initial condition $\tilde{g}$ is equivalent to a solution to the pluriclosed flow with the initial condition $\tilde{\omega}$ on M, where $\tilde{\omega}$ is the associated real $(1,1)$-form with respect to $\tilde{g}$.

This result indicates that the flow $(\mathrm{HCF})_{Q^1}$ preserves the pluriclosed condition. It is because that the pluriclosed flow preserves the pluriclosed condition and the solution to the pluriclosed flow solves $(\mathrm{HCF})_{Q^1}$ (Streets-Tian identifiability theorem), and moreover since the solutions are unique, a solution to $(\mathrm{HCF})_{Q^1}$ with the pluriclosed initial condition coincides with a solution to the pluriclosed flow with the same initial metric, and hence the flow $(\mathrm{HCF})_{Q^1}$ must preserve the pluriclosed condition. Our approach is to try to generalize their flow to almost Hermitian cases and to expect to obtain similar results as in the complex cases. We would like to define two parabolic flows and show that the identifiability theorem holds for these flows on compact almost Hermitian manifolds as well.

Let (M, J) be a compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated unique real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1} g_{r\bar{k}} \zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. We would like to define a parabolic flow of almost Hermitian metrics with an almost Hermitian initial metric ω_0 on (M, J). We will call it *the almost Hermitian flow* (AHF):

$$(\mathrm{AHF}) \quad \begin{cases} \dfrac{\partial}{\partial t} \omega(t) = \partial \partial^*_{g(t)} \omega(t) + \bar{\partial} \bar{\partial}^*_{g(t)} \omega(t) - P(\omega(t)) =: -\Phi(\omega(t)), \\ \\ \omega(0) = \omega_0, \end{cases}$$

where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators with respect to metrics $g(t)$, and $P(\omega)$ is one of the Ricci-type curvatures of the Chern curvature. One has with an arbitrary $(1,0)$-frame $\{Z_r\}$ with respect to g, $P_{i\bar{j}} = g^{k\bar{l}} \Omega_{i\bar{j}k\bar{l}}$, where Ω is the curvature of the Chern connection ∇ on (M, J, g).

Our first goal is to prove that the operator $\omega \mapsto \Phi(\omega)$ is a strictly elliptic operator for an almost Hermitian metric ω, which means that the equation (AHF) with an almost Hermitian initial metric is a strictly parabolic equation. Hence the short-time existence and the uniqueness of the solution (AHF) follows from the standard parabolic theory since the manifold is supposed to be compact. This flow (AHF) coincides with the pluriclosed flow if J is integrable and also the initial metric is pluriclosed.

Theorem 1.1. Given a compact almost Hermitian manifold (M, J, ω_0), there exists a unique solution to (AHF) with initial condition ω_0 on $[0, \varepsilon)$ for some $\varepsilon > 0$.

The following result indicates that the parabolic flow (AHCF) introduced in the statement of Theorem 1.2 could play the same role as the flow $(\mathrm{HCF})_{Q^1}$ on complex manifolds. We may expect to have some other similar results as in [12], [13] and [14] for (AHCF). Our second goal is to show the following generalized Streets-Tian identifiability theorem. We denote by S one of the Ricci-type curvatures of the Chern curvature, which is locally given by $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$.

Theorem 1.2. Let (M, J, g_0) be a compact almost Hermitian manifold with fundamental form ω_0. Then the metric g_t of the solution ω_t to (AHF) starting at ω_0 evolves as

$$\text{(AHCF)} \quad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) - Q^7(g(t)) - Q^8(g(t)) + BT'(g(t)) + \bar{Z}(T')(g(t)), \\ g(0) = g_0, \end{cases}$$

where $w_i := T_{ir\bar{r}}$,

$$BT'_{i\bar{j}} := B^{j}_{\bar{r}p}T_{ir\bar{p}} + B^{r}_{\bar{p}i}T_{pr\bar{j}} + B^{p}_{\bar{r}r}T_{pi\bar{j}} + B^{r}_{\bar{j}i}w_r, \quad \bar{Z}(T')_{i\bar{j}} := -Z_{\bar{r}}(T^s_{ri})g_{s\bar{j}} - Z_{\bar{j}}(w_i) - g^{p\bar{q}}T^r_{pi}Z_{\bar{j}}(g_{r\bar{q}}).$$

Here T is the torsion of the Chern connection and the components are computed with respect to a unitary frame. Then a solution to (AHCF) with initial condition g_0 is equivalent to a solution to (AHF) starting at the same initial condition ω_0.

In the sequel we refer to (AHCF) as *almost Hermitian curvature flow*. Note that this result in Theorem 1.2 implies that there exists a unique short-time solution to (AHCF) with initial almost Hermitian metric on a compact almost Hermitian manifold by applying Theorem 1.1. The parabolic flow (AHCF) coincides with the flow $(\mathrm{HCF})_{Q^1}$ starting at a pluriclosed metric ω_0 if J is integrable .

Proposition 1.2. The parabolic flow (AHCF) coincides with the flow $(\mathrm{HCF})_{Q^1}$ starting at a pluriclosed metric if J is integrable.

Proof. Under our assumption that J is integrable, we have $Q^7 = Q^8 = BT' = 0$. Also, since we have $\partial_{\bar{r}}T_{ri\bar{j}} = \partial_{\bar{j}}T_{ri\bar{r}}$ for a pluriclosed metric on a Hermitian manifold (one can check that (HCF) preserves the pluriclosedness (cf. [12, Theorem 3.4])) and then we may choose a local $(1,0)$-frame $Z_r = \frac{\partial}{\partial z_r}$ for some complex local coordinate $\{z_1, \ldots, z_n\}$, we obtain

$$\begin{aligned} \bar{Z}(T')_{i\bar{j}} &= -\partial_{\bar{r}}(T^s_{ri})g_{s\bar{j}} - \partial_{\bar{j}}(w_i) - g^{p\bar{q}}T^r_{pi}\partial_{\bar{j}}g_{r\bar{q}} = -\partial_{\bar{r}}T_{ri\bar{j}} + T^s_{ri}\partial_{\bar{r}}g_{s\bar{j}} - \partial_{\bar{j}}T_{ir\bar{r}} - g^{p\bar{q}}T^r_{pi}\Gamma^{\bar{k}}_{\bar{j}\bar{q}}g_{r\bar{k}} \\ &= -\partial_{\bar{j}}T_{ri\bar{r}} - \partial_{\bar{j}}T_{ir\bar{r}} + T^s_{ri}\Gamma^{\bar{k}}_{\bar{r}\bar{j}}g_{s\bar{k}} - T^s_{ri}\Gamma^{\bar{k}}_{\bar{j}\bar{r}}g_{s\bar{k}} = T_{ir\bar{s}}T_{\bar{j}\bar{r}s} = Q^1_{i\bar{j}}, \end{aligned}$$

where we are writing ∂_r, $\partial_{\bar{j}}$ for $\frac{\partial}{\partial z_r}$, $\frac{\partial}{\partial z_{\bar{j}}}$ respectively. Combining these yields the result. $\square$

The result in Proposition 1.2 tells us that our flow (AHCF) can be considered as a generalized flow of the pluriclosed flow and $(\mathrm{HCF})_{Q^1}$. Here, concerning the difference between the flow (AHCF) and Vezzoni's flow in [15], Vezzoni studied the parabolic flow on a compact almost Hermitian manifold (M^{2n}, J, g_0) such that

$$\frac{\partial}{\partial t}g(t) = -S - Q^7 - Q^8 + Q =: -K$$

with $g(0) = g_0$, where $Q^1, Q^2, Q^3, Q^4, Q^7, Q^8$ are quadratics in the torsion of the Chern connection (cf. [15, pg. 712])

$$Q^1_{i\bar{j}} := T_{ik\bar{r}}T_{\bar{j}\bar{k}r}, \quad Q^2_{i\bar{j}} := T_{\bar{k}\bar{r}i}T_{kr\bar{j}}, \quad Q^3_{i\bar{j}} := T_{ik\bar{k}}T_{\bar{j}\bar{r}r},$$

and

$$Q^4_{i\bar{j}} := \frac{1}{2}(T_{rk\bar{k}}T_{\bar{r}\bar{j}i} + T_{\bar{r}\bar{k}k}T_{ri\bar{j}}), \quad Q^7_{i\bar{j}} := T_{irk}T_{\bar{r}\bar{k}\bar{j}}, \quad Q^8_{i\bar{j}} := T_{irk}T_{\bar{j}\bar{k}\bar{r}}$$

and $Q := \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4$. These components are defined using an arbitrary unitary frame. Vezzoni's flow was defined for generalizing some studies on $(\mathrm{HCF})_Q$ in [18] and Hermitian Hilbert functional:

$$\mathbb{F}(g) = \mathrm{Vol}(M)^{\frac{1-n}{n}} \int_M k dV,$$

where $k = \mathrm{tr}_g K = s - \frac{1}{4}|T'|^2 - \frac{1}{2}|w|^2$, where s is the scalar curvature of the Chern connection. Note that we have $\mathrm{tr}_g(Q^7 + Q^8) = 0$. Vezzoni showed that a metric g is a critical point of $\mathbb{F}$ if and only if k is constant and $K = \frac{k}{n}g$.

Our approach is different from Vezzoni's one, which is to try to generalize Streets-Tian identifiability theorem in [12]. Note that we have $P = S + \mathrm{div}^{\nabla}T' - \nabla\bar{w} + Q^7 + Q^8$ for any almost Hermitian metric g (cf. [15, Lemma 3.5]), where T' is the torsion of the Chern connection ∇ associated to g, $(\mathrm{div}^{\nabla}T')_{i\bar{j}} = g^{k\bar{l}}\nabla_{\bar{l}}T_{ki\bar{j}}$, $(\nabla\bar{w})_{i\bar{j}} = g^{k\bar{l}}\nabla_i T_{\bar{j}\bar{l}k}$. For this purpose, we show that $\mathrm{div}^{\nabla}T' = -\bar{\nabla}w - BT' - \bar{Z}(T')$, $\partial\partial^*_g\omega = -\nabla\bar{w}$ for any almost Hermitian metric ω, and by using these formulae, we generalized the identifiability theorem. This generalized result indicates that (AHCF) could possibly play the same role as the flow $(\mathrm{HCF})_{Q^1}$ and the pluriclosed flow. We may expect to have some other similar results as in [12] for (AHCF).

2 An almost pluriclosed flow

Let (M, J) be a compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. We would like to define a parabolic flow of almost Hermitian metrics with an almost pluriclosed initial metric ω_0 on (M, J). We say a metric ω is almost pluriclosed if ω is an almost Hermitian metric and $\partial\bar{\partial}$-closed (cf. Definition 2.1). We will call it the almost pluriclosed flow (APF):

$$\text{(APF)} \quad \begin{cases} \frac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P(\omega(t)) =: -\Phi(\omega(t)), \\ \\ \omega(0) = \omega_0, \end{cases}$$

where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators with respect to metrics $g(t)$, and $P(\omega)$ is one of the Ricci-type curvatures of the Chern curvature. One has with an arbitrary $(1,0)$-frame $\{Z_r\}$ with respect to g, $P_{i\bar{j}} = g^{k\bar{l}}\Omega_{i\bar{j}k\bar{l}} = -g^{k\bar{l}}Z_{\bar{j}}Z_i(g_{k\bar{l}}) + \mathcal{O}(\partial\omega)$, where Ω is the curvature of the Chern connection ∇ on (M, g, J) and $\mathcal{O}(\partial\omega)$ means an expression which only depends on at most first derivatives of ω.

First, by using the almost pluriclosedness, we prove that the operator $\omega \mapsto \Phi(\omega)$ is a strictly elliptic operator for an almost pluriclosed metric ω, which means that the equation (APF) with an almost pluriclosed initial metric is a strictly parabolic equation. Hence the short-time existence and the uniqueness of the solution (APF) follows from the standard parabolic theory since the manifold is supposed to be compact. This flow (APF) coincides with the pluriclosed flow if J is integrable and also the initial metric is pluriclosed (cf. [12], [14]). We define an almost pluriclosed metric on almost Hermitian manifolds.

Definition 2.1. Let (M, J) be an almost complex manifold. A metric g is called an almost pluriclosed metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form (,which is called the fundamental $(1,1)$-form $\omega = g(J\cdot, \cdot)$ of the almost Hermitian metric g,) $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $\partial\bar{\partial}\omega = 0$.

We will also refer to the associated real $(1,1)$-form ω as an almost pluriclosed metric.

Theorem 2.1. Given a compact almost Hermitian manifold (M, J, ω_0) with almost pluriclosed metric ω_0, there exists a unique solution $\omega(t)$ to (APF) with initial condition ω_0 for $t \in [0, \varepsilon)$ for some $\varepsilon > 0$. Moreover, if $\omega(0)$ is almost pluriclosed, the metric $\omega(t)$ is almost pluriclosed for all $t \in [0, \varepsilon)$.

Since the almost pluriclosedness can be preserved along the flow (AHF), we may use the almost pluriclosedness for proving its short-time existence.

We denote by S one of the Ricci-type curvatures of the Chern curvature, which is locally given by $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$. Second, we prove that a solution of the almost pluriclosed flow with initial almost pluriclosed metric ω_0 is equivalent to a solution to (AHCF) starting at the initial almost pluriclosed metric ω_0 on a compact almost complex manifold.

Theorem 2.2. Given a compact almost Hermitian manifold (M, J, ω_0) with almost pluriclosed metric ω_0, if $(M, J, \omega(t))$ is a solution to (APF) starting at ω_0, then it coincides with a solution to (AHCF) starting at ω_0.

Third, we prove that the equation (AHCF) is a strictly parabolic equation. Therefore, the short-time existence and the uniqueness of the solution to (AHCF) with initial condition ω_0 follow from the standard parabolic theory since the manifold is supposed to be compact. Since the solution to (AHCF) are unique, the solution to (AHCF) exactly coincides with the solution to (APF).

Theorem 2.3. Given a compact almost Hermitian manifold (M, J, ω_0) with almost pluriclosed metric ω_0, there exists a unique solution $\omega(t)$ to (AHCF) with initial condition ω_0 for $t \in [0, \varepsilon)$ for some $\varepsilon > 0$. Moreover, if $\omega(0)$ is almost pluriclosed, the metric $\omega(t)$ is almost pluriclosed for all $t \in [0, \varepsilon)$.

Now we show that the almost pluriclosedness can be preserved along the solution to (APF). We need the following lemma.

Lemma 2.1. One has

$$\partial\bar{\partial}\partial = 0, \quad \partial\bar{\partial}^2 = 0.$$

Proof. Using $A\bar{\partial} + \partial^2 + \bar{\partial}A = 0$, $A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0$, $\partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0$, then we compute

$$\begin{aligned} \partial\bar{\partial}\partial &= -\partial(A\bar{A} + \partial\bar{\partial} + \bar{A}A) = A\partial\bar{A} - \partial^2\bar{\partial} - \partial\bar{A}A = A\partial\bar{A} + A\bar{\partial}^2 + \bar{\partial}A\bar{\partial} - \partial\bar{A}A \\ &= A\partial\bar{A} - A\partial\bar{A} - A\bar{A}\partial + \bar{\partial}A\bar{\partial} - \partial\bar{A}A = -A\bar{A}\partial + \bar{\partial}A\bar{\partial} - \partial\bar{A}A. \end{aligned}$$

Since $A^2 = 0$, we obtain $\partial\bar{\partial}\partial A = \bar{\partial}A\bar{\partial}A - A\bar{A}\partial A - \partial\bar{A}AA = \bar{\partial}A\bar{\partial}A - A\bar{A}\partial A$, equivalently we obtain

$$(\partial\bar{\partial}\partial - \bar{\partial}A\bar{\partial} + A\bar{A}\partial)A = 0,$$

which implies that we have

$$\partial\bar{\partial}\partial = \bar{\partial}A\bar{\partial} - A\bar{A}\partial, \quad \partial\bar{A}A = 0.$$

And then we compute $\partial\bar{\partial}\partial = \bar{\partial}A\bar{\partial} - A\bar{A}\partial = \bar{\partial}A\bar{\partial} + \bar{A}A\partial + \bar{\partial}\partial^2 + \partial\bar{\partial}\partial$, which tells us that we have

$$\bar{\partial}A\bar{\partial} = -\bar{A}A\partial - \bar{\partial}\partial^2.$$

From the equality $\partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0$, we obtain

$$0 = \partial\bar{A}A + \bar{\partial}^2 A + \bar{A}\partial A = \bar{\partial}^2 A + \bar{A}\partial A,$$

where we used that $\partial\bar{A}A = 0$. Then we have $\partial\bar{A}\partial A + \partial\bar{\partial}^2 A = (\partial\bar{A}\partial + \partial\bar{\partial}^2)A = 0$, which implies that we have

$$\partial\bar{A}\partial = -\partial\bar{\partial}^2, \quad \bar{\partial}A\bar{\partial} = -\bar{\partial}\partial^2, \quad \bar{A}A\partial = 0, \quad \bar{\partial}^2 A = 0$$

and also we obtain from $\bar{\partial}^2 A + \bar{A}\partial A = 0$,

$$A\bar{\partial}^2 A + A\bar{A}\partial A = 0,$$

equivalently

$$(A\bar{\partial}^2 + A\bar{A}\partial)A = 0,$$

which gives us that

$$A\bar{\partial}^2 = -A\bar{A}\partial.$$

Therefore, $\partial\bar{\partial}\partial = \bar{\partial}A\bar{\partial} - A\bar{A}\partial = -\bar{\partial}\partial^2 + A\bar{\partial}^2$, which gives

$$\partial\bar{\partial}\partial A = -\bar{\partial}\partial^2 A + A\bar{\partial}^2 A = -\bar{\partial}\partial^2 A,$$

equivalently $(\partial\bar{\partial}\partial + \bar{\partial}\partial^2)A = 0$, which implies that

$$\partial\bar{\partial}\partial = -\bar{\partial}\partial^2, \quad A\bar{A}\partial = 0.$$

Finally, we obtain since $\partial^2\bar{A} = 0$ and $A\bar{A}\partial = \bar{A}A\partial = 0$,

$$\partial\bar{\partial}\partial\bar{A} = -\bar{\partial}\partial^2\bar{A} - A\bar{A}\partial\bar{A} - \bar{A}A\partial\bar{A} = (-A\bar{A}\partial - \bar{A}A\partial)\bar{A} = 0 \cdot \bar{A} = 0,$$

which implies we have

$$\partial\bar{\partial}\partial = 0, \quad \partial\bar{\partial}^2 = 0.$$

□

Proposition 2.1. The almost pluriclosedness is preserved along the solution to (APF).

PROOF. From the results in Lemma 3.4, a direct computation gives

$$\frac{\partial}{\partial t}\partial\bar{\partial}\omega(t) = -\partial\bar{\partial}\Phi(\omega(t)) = \partial\bar{\partial}\partial\partial^*_{g(t)}\omega(t) + \partial\bar{\partial}\bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - \partial\bar{\partial}P(\omega(t)) = 0,$$

where we used that $P(\omega)$ is closed. □

3 A scalar Calabi-type flow

In [2], Bedulli and Vezzoni introduced a geometric flow of Hermitian metrics which evolves an initial metric along the second derivative of the Chern scalar curvature. They showed that the flow has a unique short-time solution and provide a stability result when the background metric is Kähler with constant scalar curvature. We extend thier results in the Hermitian geometry to the almost Hermitian geometry.

Let (M, J) be a compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$.

Given an almost Hermitian form ω on a compact almost complex manifold (M, J), we consider the set

$$C^\infty_\omega(M) := \{\varphi \in C^\infty(M) | \omega^{n-1} + \sqrt{-1}\partial\bar{\partial}(\varphi\omega^{n-2}) > 0\},$$

where the $\sqrt{-1}\partial\bar{\partial}$ is in the almost Hermitian geometry, given by for $\varphi \in C^\infty(M)$,

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi\zeta^i \wedge \zeta^{\bar{j}}.$$

Every $\varphi \in C^\infty_\omega(M)$ induces an almost Hermitian form ω_φ defined by

$$\omega_\varphi^{n-1} = \omega^{n-1} + \sqrt{-1}\partial\bar{\partial}(\varphi\omega^{n-2}).$$

We denote by $\mathcal{C}_\omega(M)$ the set

$$\mathcal{C}_\omega(M) := \{\omega_\varphi | \varphi \in C^\infty_\omega(M)\}.$$

We investigate the solutions $\omega(t) \in \mathcal{C}_\omega(M)$ to the following parabolic flow;

$$(\dagger) \quad \begin{cases} \dfrac{\partial}{\partial t}\omega(t)^{n-1} = \sqrt{-1}\partial\bar{\partial}(s_{\omega(t)}\omega^{n-2}), \\ \\ \omega(0) = \omega_0, \end{cases}$$

whose definition depends on the background almost Hermitian form ω, where ω_0 is an almost Hermitian form and $s_{\omega(t)}$ is the Chern scalar curvature of the Chern connection induced by $\omega(t)$. If the background almost Hermitian metric is quasi-Kähler, then almost Hermitian metrics with constant Chern scalar curvature are stationary solutions to the flow ($\dagger$). A quasi-Kähler structure is an almost Hermitian structure whose associated real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$ (cf. [3]). A quasi-Kähler manifold with J integrable is a Kähler manifold.

The equation ($\dagger$) can be reduced to the scalar equation;

$$(\dagger)' \quad \begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = s_{\varphi(t)}, \\ \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\varphi_0 \in C^\infty_\omega(M)$ is such that $\omega_{\varphi_0} = \omega_0$, and for $\varphi \in C^\infty_\omega(M)$, s_φ is the Chern scalar curvature of ω_φ. If $\varphi(t)$ satisfies $(\dagger)'$, then $\omega_{\varphi(t)}$ satisfies ($\dagger$). Conversely, if $\omega_{\varphi(t)}$ satisfies ($\dagger$), then there exists $\hat{\varphi}(t)$ satisfying $\omega_{\hat{\varphi}(t)} = \omega_{\varphi(t)}$ and $(\dagger)'$.

We next define an almost balanced metric.

Definition 3.1. Let (M^{2n}, J) be an almost complex manifold. A metric g is called an almost balanced metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d(\omega^{n-1}) = 0$. And when an almost Hermitian metric g is almost balanced, the triple (M^{2n}, J, g) will be called an almost balanced manifold.

We will show that the flow ($\dagger$) preserves the almost balanced condition. We obtain the following generalized result in the almost Hermitian geometry.

Theorem 3.1. The flow ($\dagger$) has always a unique short-time solution $\{\omega(t)\}_{t\in[0,T_{\max})}$. If the initial almost Hermitian metric ω_0 satisfies almost balanced condition, then the metrics $\omega(t)$ are almost balanced for all $t \in [0, T_{\max})$. Assume that the background almost Hermitian metric ω is quasi-Kähler with constant Chern scalar curvature. Then if ω_0 is close enough to ω in C^∞-topology, the solution $\{\omega(t)\}_t$ is defined for any $t \in [0, \infty)$, and converges in C^∞-topology to ω as $t \to \infty$.

Let (M, g) be an oriented compact Riemannian manifold with the volume form dV_g with respect to g and let $W^{2r,2}_+(M)$ be an open neighborhood of 0 in $W^{2r,2}(M)$ which is invariant by additive constants. For $k > 2r$, we define $W^{k,2}_+(M) := W^{2r,2}_+(M) \cap W^{k,2}(M)$ and $C^\infty_+(M) := C^\infty(M) \cap W^{2r,2}_+(M)$. Let $Q : W^{2r,2}_+(M) \to L^2(M)$ be a smooth *elliptic* operator of order $2r$ in a strong sense explained such as in [2], [4] and [9] (we will see later), and we denote by L the differential of Q at 0. Assume further that Q satisfies the following conditions:

(I) $Q(0) = 0$ and $Q(\psi) = Q(\psi + a)$ for every $\psi \in W^{2r,2}_{+}(M)$, $a \in \mathbb{R}$ (the set of constant functions is identified with $\mathbb{R}$);

(II) The kernel of L is made only by constant functions and $L(W^{2r,2}_0(M)) \subseteq L^2_0(M)$, where the subscript 0 means that the elements have average 0 with respect to g;

(III) L is symmetric and semi-negative definite with respect to the L^2-scalar product induced by the fixed metric g on M, i.e., for every $\psi_1, \psi_2 \in W^{2r,2}(M)$,

$$\int_M L(\psi_1)\psi_2 dV_g = \int_M L(\psi_2)\psi_1 dV_g \quad \text{and} \quad \int_M L(\psi_1)\psi_1 dV_g \leq 0.$$

We note here the ellipticness *in a strong sense* for the operator Q as we mentioned before. Let (M, g) be a compact m-dimensional Riemannian manifold and let $Q : C^\infty(M) \to C^\infty(M)$ be a quasi-linear partial differential operator of order $2r$. Then, $Q(\psi)$ is locally given as

$$\begin{aligned} Q(\psi)(x) &= A^{i_1 \dots i_{2r}}(x)(\psi(x), \nabla\psi(x), \dots, \nabla^{2r-1}\psi(x))\nabla^{2r}_{i_1 \dots i_{2r}}\psi(x) \\ &\quad + b(x)(\psi(x), \nabla\psi(x), \dots, \nabla^{2r-1}\psi(x)), \end{aligned}$$

where ∇ is the Levi-Civita connection of g and the functions $A^{i_1 \dots i_{2r}}$ and b are smooth in their entries. We assume that Q is *elliptic* by requiring

$$A^{i_1 j_1 \dots i_r j_r} = (-1)^{r-1} E_1^{i_1 j_1} \dots E_r^{i_r j_r},$$

for some $(2, 0)$-tensors $E_1, \dots, E_r$ such that there exits a positive constant $\lambda \in \mathbb{R}$ such that each tensor E_k $(k = 1, \dots, r)$ satisfies

$$E_k^{ij}(x)(v_1, \dots, v_{2r-1})\xi_i\xi_j \geq \lambda |\xi|_g^2 \quad \text{for every } \xi \in T_x^*M,$$

when $x \in M$ and $v_k \in \otimes_k T_x^*M$. Given such a smooth *elliptic* quasi-linear operator Q of order $2r$ and an initial datum $\varphi_0 \in C^\infty(M)$, we consider the parabolic equation

$$(\sharp) \quad \begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = Q(\varphi(t)), \\[2ex] \varphi(0) = \varphi_0. \end{cases}$$

We notice that the following result can be found in [4] and [9].

Proposition 3.1. (cf. [4, Theorem 7.15], [9, Theorem 1.1]) For every $\varphi_0 \in C^\infty(M)$, there exists a time $T_{\max} > 0$ such that the parabolic problem $(\sharp)$ has a maximal solution $\varphi \in C^\infty(M \times [0, T_{\max}))$. Moreover, the solution φ is unique and depends continuously on the initial datum φ_0 in the $C^\infty(M)$-topology.

Under the assumptions (I), (II) and (III) for the smooth *elliptic* operator Q, Bedulli and Vezzoni obtained the following stability result:

Proposition 3.2. ([2, Theorem 1.2]) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varphi_0 \in C^\infty_+(M)$ satisfies $||\varphi_0||_{C^\infty} < \delta$, then the parabolic problem

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = Q(\varphi(t)), \\ \\ \varphi(0) = \varphi_0 \end{cases}$$

has a unique solution $\varphi(t) \in C^\infty(M \times [0, \infty))$ such that $\varphi(t) \in C^\infty_+(M)$ for every t and satisfies

(I) $||\varphi(t)||_{C^\infty} < \varepsilon$ for every $t \in [0, \infty)$;

(II) $\varphi(t)$ converges in C^∞-topology to a smooth function φ_∞ such that $Q(\varphi_\infty) = 0$.

We need the following lemma for proving the preservation of the almost balancedness. By applying some results in the proof of Lemma 2.1, we obtain the following equalities.

Lemma 3.1. One has

$$\partial^2\bar{\partial} = 0, \quad \bar{\partial}\partial\bar{\partial} = 0.$$

Proof. By using $A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0$, $\partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0$, $\partial A + A\partial = 0$ and the results in the proof of [9, Lemma 3.4]; $\partial\bar{A}A = 0$, $A\bar{\partial}^2 = -A\bar{A}\partial$ and $\partial\bar{\partial}\partial = 0$, we obtain

$$\begin{aligned} \partial^2\bar{\partial} = \partial\partial\bar{\partial} &= -\partial(A\bar{A} + \bar{\partial}\partial + \bar{A}A) = -\partial A\bar{A} - \partial\bar{\partial}\partial - \partial\bar{A}A \\ &= A\partial\bar{A} = -A\bar{\partial}^2 - A\bar{A}\partial = 0. \end{aligned}$$

□

By checking conditions (I), (II) and (III) for the elliptic operator $Q(\psi) = s_\psi - R$, $\psi \in C^\infty_\omega(M)$, where R is the constant Chern scalar curvature of the quasi-Kähler metric ω, and applying Proposition 3.1, 3.2, we can show Theorem 3.1.

4 A parabolic flow of almost balanced metrics

In [1], Bedulli and Vezzoni introduced a parabolic flow of balanced metrics and they showed that the flow has a unique short-time solution. We extend their results in the Hermitian geometry to the almost Hermitian geometry.

Let M^n be a complex manifold and let ω be a fundamental $(1,1)$-form associated to a Hermitian metric g on M. In the Kähler geometry, the Calabi flow is a well-known gradient flow of the Calabi functional

$$\omega \mapsto \int_M (s_\omega)^2 \frac{\omega^n}{n!}$$

as it is restricted to the cohomology class of an initial Kähler metric ω_0

$$C_{\omega_0} = \{\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0 | \phi \in C^\infty(M, \mathbb{R})\}$$

where s_ω is the scalar curvature of the metric ω. The Calabi flow is as follows:

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = \sqrt{-1}\partial\bar{\partial}s_{\omega(t)}, \\ \\ \omega(0) = \omega_0. \end{cases}$$

The flow above can be alternatively expressed in terms of positive $(n-1, n-1)$-forms as

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \sqrt{-1}\partial\bar{\partial} *_t (P_t \wedge *_t\varphi(t)), \\ \\ \varphi(0) = \varphi_0, \end{cases}$$

where P_t is the Ricci form of $\varphi(t)$ and $\varphi_0 = *_{\omega_0}\omega_0$.

Let (M^{2n}, J) be a real $2n$-dimensional compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$.

Let φ_0 be a positive closed $(n-1, n-1)$-form on M. We investigate the following parabolic flow of almost balanced structures φ on M;

$$(\ddagger) \quad \begin{cases} \frac{\partial}{\partial t}\varphi(t) = \sqrt{-1}\partial\bar{\partial} *_t (P_t \wedge *_t\varphi(t)) + (n-1)\Delta_{BC}\varphi(t), \\ \\ d\varphi(t) = 0, \\ \\ \varphi(0) = \varphi_0, \end{cases}$$

where

$$\Delta_{BC} := \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial$$

is the modified Bott-Chern Laplacian (cf. [10]) and $*_t$ and P_t are the Hodge star operator and the Chern-Ricci form locally given by $P_{i\bar{j}} = g^{k\bar{l}}\Omega_{i\bar{j}k\bar{l}}$, Ω is the curvature form of the Chern connection with respect to g.

We can show that the modified Bott-Chern Laplacian Δ_{BC} is a fourth order elliptic operator by considering the symbol of the operator in the base $(\zeta_I \wedge \overline{\zeta_J})_{|I|=p,|J|=q}$, where $\{\zeta_I\}_I$ is the associated coframe with respect to the local $(1,0)$-frame $\{Z_I\}_I$ around a point $x \in M$ as in [10].

We introduce the definition of an almost balanced metric.

Definition 4.1. Let (M^{2n}, J) be an almost complex manifold. A metric g is called an almost balanced metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d(\omega^{n-1}) = 0$. And when an almost Hermitian metric g is almost balanced, the triple (M^{2n}, J, g) will be called an almost balanced manifold.

An almost balanced structure can be alternatively regarded as a closed positive real $(n-1, n-1)$-form φ. We define the Bott-Chern cohomology in almost complex geometry as follows:

$$H_{BC}(M) = \frac{\ker d}{\mathrm{Im}(\partial\bar{\partial})}.$$

Our main result is as follows.

Theorem 4.1. Let (M, J, g) be a compact almost Hermitian manifold and let φ_0 be a closed positive real $(n-1, n-1)$-form on M. The flow (†) admits a unique solution in the Bott-Chern class of $[\varphi_0]$ defined in a maximal interval $[0, \varepsilon)$ on M. Moreover, if the initial structure is Kähler, then (†) reduces to the Calabi flow.

Let Δ_A denote the modified Aeppli Laplacian (cf. [10]), which is defined by

$$\Delta_A := \bar{\partial}^* \partial^* \partial \bar{\partial} + \partial \bar{\partial} \bar{\partial}^* \partial^* + \partial \bar{\partial}^* \bar{\partial} \partial^* + \bar{\partial} \partial^* \partial \bar{\partial}^* + \partial \partial^* + \bar{\partial} \bar{\partial}^*.$$

We can show that the modified Aeppli Laplacian Δ_A is a fourth order elliptic operator in the same way for proving that Δ_{BC} is elliptic. We will need a Hodge-like decomposition induced by Δ_A. We can prove the Aeppli decomposition based on the result that Δ_A is elliptic.

Proposition 4.1. (cf. [10]) If (M, J, g) is a compact almost Hermitian manifold, then we have the following orthogonal decomposition for every (p, q)

$$C^\infty(M, \Lambda^{p,q}) = \mathcal{H}^{p,q}_{\Delta_A}(M) \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}) \oplus \mathrm{Im}(\partial\bar{\partial})^*,$$

where $\mathcal{H}^{p,q}_{\Delta_A}(M) = \mathrm{Ker}\Delta_A$.

The following proposition is a crucial step in the proof of Theorem 4.1. The proof is the similar to the one in [1, Proposition 2.4]. The difference appears only in the part of using Lemma 2.1 and 3.1.

Proposition 4.2. (cf. [1, Proposition 2.4]) Let G_A be the Green operator associated to the modified Aeppli Laplacian Δ_A. Then for every $\psi \in \partial\bar{\partial}C^\infty(M, \Lambda^{pq})$, we have

$$\psi = \partial\bar{\partial} G_A (\partial\bar{\partial})^*(\psi).$$

Proof. Choose arbitrary $\psi \in \partial\bar{\partial}\Lambda^{p,q}$. By applying the Aeppli decomposition and $u \in \mathcal{H}^{p,q}_{\Delta_A}(M) \Leftrightarrow \bar{\partial}^* u = \partial^* u = \partial\bar{\partial} u = 0$ (cf. [10]), and $\partial\bar{\partial}\partial = 0$, $\partial\bar{\partial}^2 = 0$ in Lemma 2.1, we have $\psi = \partial\bar{\partial}\beta$ with $\beta \in \mathrm{Im}(\partial\bar{\partial})^*$. Especially, since we have $\partial^*\bar{\partial}^*\partial^* = 0$, $\bar{\partial}^*\bar{\partial}^*\partial^* = 0$ from Lemma 2.1 and 3.1, we have $\Delta_A \beta = (\partial\bar{\partial})^*\partial\bar{\partial}\beta$, which tells us that $\beta = G_A((\partial\bar{\partial})^*\partial\bar{\partial}\beta) = G_A((\partial\bar{\partial})^*\psi)$, and hence we obtain $\psi = \partial\bar{\partial}G_A(\partial\bar{\partial})^*(\psi)$. □

Next we introduce a Hodge system.

Definition 4.2. A Hodge system on a manifold M consists of the following sequaence

$$\begin{array}{ccc} C^\infty(M, E_-) & \xrightarrow{D} & C^\infty(M, E) \\ \Delta_D \big\downarrow & & \\ C^\infty(M, E_-) & \xleftarrow[D^*]{} & C^\infty(M, E) \end{array}$$

where E_- and E are fiber bundles over M with an assigned metric along their fibers, D is a differential operator, D^* is the formal adjoint of D and Δ_D is an elliptic operator such that $\psi = DGD^*\psi$ for every $\psi \in \mathrm{Im}D$, where G is the Green operator of Δ_D.

Consider a Hodge system on a compact manifold M as in Definition 2.1. Let O be an open subset of E such that $\pi(O) = M$, where $\pi : E \to M$ is the projection. Consider a non-linear partial differential operator of order $2m$

$$L : C^\infty(M, O) \to C^\infty(M, E)$$

and a fixed initial datum $\varphi_0 \in C^\infty(M, O)$ such that

$$L(\varphi_0 + D\gamma) \in \mathrm{Im}D$$

for every $\gamma \in C^\infty(M, E_-)$. We consider the following evolution problem

$$(\mathrm{II}) \quad \begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = L(\varphi(t)), \\ \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\varphi(t)$ is in the following space

$$U = \{\varphi_0 + D\gamma | \gamma \in C^\infty(M, E_-)\} \cap C^\infty(M, O)$$

and $\varphi(t)$ is required to depend smoothly on time.

Let $\mathcal{D}^{2m}(E, E)$ denote the space of partial differential operators on E of order $\leq 2m$, which can be seen as the space of smooth sections of a vector bundle. A linear partial operator Q of order $2m$ is said to be *strongly elliptic* if its principal symbol $\sigma_Q(x, \xi)$ satisfies the following inequality:

$$-\langle \sigma_Q(x, \xi)v, v\rangle_E \geq \lambda|\xi|^{2m}|v|^{2m}$$

for some positive constant λ and for all $(x, \xi) \in TM$, $\xi \neq 0$ and $v \in E_x$, whose definition dose not depend on the metric $\langle \cdot, \cdot \rangle_E$ along the fibers onf E. The principal symbol of Q is defined by

$$\sigma_Q(x, \xi)v = \frac{(\sqrt{-1})^{2m}}{(2m)!} Q(f^{2m}u)(x)$$

for $f \in C^\infty(M)$ with $f(x) = 0$, $d_x f = \xi$ and $u \in C^\infty(E)$ with $u(x) = v$. We denote by $L_*|_\varphi$ the derivative of the operator L at φ.

Theorem 4.2. ([1, Theorem 3.2]) Let (E_-, E, D, Δ_D) be a Hodge system on a compact Riemannian manifold M. Let L, φ_0 and U be as above. Assume that there exists a nonlinear partial differential operator

$$\tilde{L} : C^\infty(M, O) \to \mathcal{D}^{2m}(E, E), \quad \varphi \mapsto \tilde{L}_\varphi$$

such that

(I) $\tilde{L}_\varphi$ is strong elliptic for every $\varphi \in U$;

(II) $L_*|_\varphi(\psi) = \tilde{L}_\varphi(\psi)$ for every $\varphi \in U$ and $\psi \in DC^\infty(M, E_-)$.

Assume further that

$$L_*|_\varphi(D\theta) = Dl_\varphi(\theta)$$

for every $\theta \in C^\infty(M, E_-)$, where l_φ is a strongly elliptic linear differential operator on E_-. Then there exists $\varepsilon > 0$ such that the system (II) has a unique solution $\varphi \in C^\infty([0,\varepsilon), U)$.

We consider

$$E_- = \Lambda_{\mathbb{R}}^{n-2,n-2}M \xrightarrow{D=\sqrt{-1}\partial\bar{\partial}} E = \Lambda_{\mathbb{R}}^{n-1,n-1}M,$$

where $\Lambda_{\mathbb{R}}^{p,p}M$ is the bundle of real (p,p)-forms; the subset U is the set of smooth sections of $\Lambda_+^{n-1,n-1}$ lying in the same cohomology class as φ_0. Let

$$L : C^\infty(M, \Lambda_+^{n-1,n-1}M) \to C^\infty(M, \Lambda_{\mathbb{R}}^{n-1,n-1}M)$$

be the operator $L(\varphi) = \sqrt{-1}\partial\bar{\partial} * (P \wedge *\varphi) + (n-1)\Delta_{BC}\varphi$.

We can show that for every closed $\varphi \in C^\infty(M, \Lambda_+^{n-1,n-1}M)$ and every closed $\psi \in C^\infty(M, \Lambda_{\mathbb{R}}^{n-1,n-1}M)$, we have

$$L_*|_\varphi(\psi) = (1-n)\Delta_{BC}\psi + \sqrt{-1}\partial\bar{\partial}\Psi_\varphi(\psi),$$

where Ψ_φ is a linear algebraic operator on ψ with coefficients depending on the torsion of φ in a universal way. Let us consider

$$l_\varphi = -(n-1)(\Delta_A)_\varphi + \sqrt{-1}\Phi_\varphi \circ \partial\bar{\partial}.$$

As we have confirmed that $-\Delta_A$ is strongly elliptic. In addition, we have the following:

Proposition 4.3. One has

$$\Delta_{BC}\partial\bar{\partial} = \partial\bar{\partial}\Delta_A.$$

Proof of Proposition 4.3. By using Lemma 3.1, we obtain

$$\begin{aligned}
\Delta_{BC}\partial\bar{\partial} &= (\partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial)\partial\bar{\partial} \\
&= \partial\bar{\partial}\bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial^*\partial\bar{\partial}\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial}\partial\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial^2\bar{\partial} + \bar{\partial}^*\bar{\partial}\partial\bar{\partial} + \partial^*\partial^2\bar{\partial} \\
&= \partial\bar{\partial}\bar{\partial}^*\partial^*\partial\bar{\partial}.
\end{aligned}$$

On the other hand, using Lemma 2.1 we have

$$\begin{aligned}
\partial\bar{\partial}\Delta_A &= \partial\bar{\partial}(\bar{\partial}^*\partial^*\partial\bar{\partial} + \partial\bar{\partial}\bar{\partial}^*\partial^* + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\partial^* + \bar{\partial}\bar{\partial}^*) \\
&= \partial\bar{\partial}\bar{\partial}^*\partial^*\partial\bar{\partial} + \partial\bar{\partial}\partial\bar{\partial}\bar{\partial}^*\partial^* + \partial\bar{\partial}\partial\bar{\partial}^*\bar{\partial}\partial^* + \partial\bar{\partial}^2\partial^*\partial\bar{\partial}^* + \partial\bar{\partial}\partial\partial^* + \partial\bar{\partial}^2\bar{\partial}^* \\
&= \partial\bar{\partial}\bar{\partial}^*\partial^*\partial\bar{\partial}.
\end{aligned}$$

Hence we get $\Delta_{BC}\partial\bar{\partial} = \partial\bar{\partial}\Delta_A$. □

Then we have that $L_*|_\varphi(D\psi) = Dl_\varphi(\psi)$ for every closed $\psi \in C^\infty(M, \Lambda_{\mathbb{R}}^{n-2,n-2}M)$. By applying Theorem 4.2, we obtain the desired unique existence result.

If φ_0 is the $(n-1,n-1)$-positive form of a Kähler structure, then the solution $\varphi_0+\beta(t)$ to (‡) corresponds to a family of Kähler forms $\omega(t)$ solving the Calabi flow (cf. [1]).

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