6次元球面のラグランジュ部分多様体と 結合的グラスマン多様体

Lagrangian submanifolds of S^6 and the associative Grassmann manifold

Kazumi Tsukada *

It is well-known that a six-dimensional sphere S^6 admits an almost complex structure defined by its natural inclusion in the space $\mathrm{Im}\mathbb{O}$ of imaginary octonions. This almost complex structure is not integrable but is nearly Kähler with respect to the induced Riemannian metric from the inner product in $\mathrm{Im}\mathbb{O}$.

An oriented three-dimensional subspace of $\mathrm{Im}\mathbb{O}$ is said to be associative if it is a canonically oriented imaginary part of some quaternion subalgebra of $\mathbb{O}$. The set of all associative subspaces is called the associative Grassmann manifold, which is denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Then it is known that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is an eight-dimensional compact symmetric quaternionic Kähler manifold.

We focus on Lagrangian submanifolds of S^6 and study the relationship of such submanifolds with the geometry of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.
This is a joint work with K.Enoyoshi ([3]).

§1 The algebra of octonions and the Lie group G_2

In this section, we recall fundamental properties of octonions following R. Harvey and H. B. Lawson [5]. Let $\mathbb{H} = \{x1+yi+zj+wk \mid x, y, z, w \in \mathbb{R}\} \cong \mathbb{R}^4$ ($i^2 = j^2 = k^2 = -1, ij = -ji = k$) be the algebra of quaternions and $Sp(1)$ the group of unit quaternions. We denote the subspace of imaginary quaternions by $\mathrm{Im}\mathbb{H}$. The algebra $\mathbb{O}$ of octonions is a normed algebra whose the multiplication is given by

$$(a + b\varepsilon)(c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon \qquad a, b, c, d \in \mathbb{H}$$

([5]). Here $\bar{a}$ is the quaternionic conjugation for $a \in \mathbb{H}$. The algebra $\mathbb{O}$ is neither commutative nor associative. Let $\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \oplus \mathbb{H}\varepsilon$ be the subspace of all imaginary parts of octonions, which is identified with seven-dimensional Euclidian space $\mathbb{R}^7$.

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We define the alternating trilinear form φ on $\mathrm{Im}\mathbb{O}$ by

$$\varphi(x, y, z) = \langle x, yz \rangle.$$

The 3-form φ is called the *associative calibration* on $\mathrm{Im}\mathbb{O}$ ([5] p.113 Definition 1.5). The Lie group G_2 is defined by

$$G_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in GL(\mathbb{O}) |\ g(xy) = g(x)g(y),\ \text{for any } x, y \in \mathbb{O}\}.$$

It is well-known that the Lie group G_2 is 14-dimensional and simple ([5]). Every automorphism of $\mathbb{O}$ fixes the subspace $\mathbb{R} \cdot 1 \subset \mathbb{O}$ and leaves the subspace $\mathrm{Im}\mathbb{O}$ invariant. We also have the facts that G_2 is a subgroup of $SO(\mathrm{Im}\mathbb{O}) \cong SO(7)$ and that the following holds:

$$G_2 = \{g \in O(7) |\ g^*\varphi = \varphi\}.$$

For the pair of unit quaternions $(q_1, q_2) \in Sp(1) \times Sp(1)$ and $a + b\varepsilon \in \mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$, we set

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a q_1^{-1} + (q_2 b q_1^{-1})\varepsilon.$$

Then we see that $\rho(q_1, q_2)$ belongs to G_2 and the kernel of ρ is $\{\pm(1, 1)\}$. Hence ρ is an action of $Sp(1) \times Sp(1)/\{\pm(1, 1)\} \cong SO(4)$ on $\mathbb{O}$.

§2 The associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ and its tangent space

We denote by $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ the Grassmann manifold of all oriented three-dimensional subspaces in $\mathrm{Im}\mathbb{O}$ with $\dim \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) = 12$. $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is isomorphic to the Riemannian symmetric space $SO(7)/SO(3) \times SO(4)$. If $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is the canonically oriented imaginary part of some quaternion subalgebra of $\mathbb{O}$, then ζ is said to be an *associative subspace*. The set of all associative subspaces is called the *associative Grassmann manifold* denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Then it is known that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is an eight-dimensional compact symmetric quaternionic Kähler manifold which is described as $G_2/SO(4)$ (cf. [5],[11]). Moreover we see that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is a totally geodesic submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$.

The associative calibration φ can be viewed as a function on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. The following has been shown by Harvey and Lawson;

Proposition 2.1 ([5]) For $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$, $\varphi(\zeta) \leq 1$ with the equality if and only if ζ is associative.

Then we define level sets $\tilde{M}(t)$ of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ for $-1 \leq t \leq 1$ by

$$\tilde{M}(t) = \{\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) \mid \varphi(\zeta) = t\}.$$

The Lie group G_2 acts transitively on each $\tilde{M}(t)$ $(-1 \leq t \leq 1)$. The level set $\tilde{M}(1)$ coincides with the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Reversing the orientation of subspaces in $\tilde{M}(-1)$, we see that $\tilde{M}(-1)$ is isometric to $\tilde{M}(1)$. For $-1 < t < 1$, $\tilde{M}(t)$ is diffeomorphic to $G_2/SO(3)$.

We will describe the tangent space of the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ following F. Nakata ([9]). By the standard argument using the theory of vector bundles for the differential geometry of Grassmann manifolds, the tangent space of the Grassmann manifold at $\xi \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is identified with the space $\mathrm{Hom}(\xi, \xi^\perp)$ of linear homomorphisms of ξ to $\xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of ξ in $\mathrm{Im}\mathbb{O}$. The tangent space of the associative Grassmann manifold $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is described as follows:

$$T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \simeq \{\gamma \in \mathrm{Hom}(\xi, \xi^\perp) \mid \gamma(e_1)e_1 + \gamma(e_2)e_2 + \gamma(e_3)e_3 = 0\},$$

where $\{e_1, e_2, e_3\}$ denotes an orthonormal basis of ξ. This description is due to Nakata ([9]). We denote the right hand side in the above equation by $\mathrm{Hom}_{ass}(\xi, \xi^\perp)$.

§3 The nearly Kähler structure and submanifolds of S^6

Let S^6 be the unit sphere in $\mathrm{Im}\mathbb{O}$ centered at the origin. Then S^6 has an almost complex structure J at $p \in S^6$ defined by

$$J_p x = px, \quad x \in T_p S^6.$$

It is known that this almost complex structure J is not integrable. The Riemannian metric on S^6 is induced from the inner product $\langle , \rangle$ on $\mathrm{Im}\mathbb{O} \cong \mathbb{R}^7$. The induced metric $\langle , \rangle$ is Hermitian with respect to J. Then we define the Kähler form ω of S^6 at $p \in S^6$ by

$$\omega_p(x, y) = \langle J_p x, y\rangle, \quad x, y \in T_p S^6.$$

Then $(S^6, J, \langle , \rangle)$ is an almost Hermitian manifold. Moreover it satisfies $(\tilde{\nabla}_X J)X = 0$ for all vector fields X on S^6 , where $\tilde{\nabla}$ denotes the Riemannian connection. An almost Hermitian manifold with this property is called a *nearly Kähler manifold*. We have the fact that the group of automorphisms of $(S^6, J, \langle , \rangle)$ is isomorphic to G_2.

The nearly Kähler six-sphere has two typical classes of submanifolds: namely the class of almost complex submanifolds and that of Lagrangian submanifolds or three-dimensional totally real submanifolds. A. Gray ([4]) showed that there do not exist four-dimensional almost complex submanifolds in S^6. R. L. Bryant([1]) proved that every compact Riemann surface can be realized as an almost complex curve in S^6. Many researchers have studied two-dimensional almost complex submanifolds or almost complex curves and obtained fruitful results.

We recall Lagrangian submanifolds of six-sphere. We define a trilinear form ϕ and a complex-valued form Ω on S^6 using the associative calibration φ as follows:

$$\begin{aligned} \phi(x, y, z) &= \varphi(x, y, J_p z) \qquad x, y, z \in T_p S^6 \\ \Omega(x, y, z) &= \phi(x, y, z) + \sqrt{-1}\,\varphi(x, y, z) \end{aligned}$$

Then Ω is a $(3,0)$-form with respect to J. We have the following interesting relations between the Kähler form and the associative calibration.

Proposition 3.1 (cf. K.Mashimo [8]) For $p \in S^6$,
(1) $\omega_p = p \rfloor \varphi$
(2) $d\omega = 3\varphi|_{S^6}$
(3) $d\phi = 4\omega \wedge \omega$

We state the special Lagrangian geometry of S^6 with respect to Ω. A three-dimensional subspace ζ of T_pS^6 is called a *Lagrangian subspace* if it holds that $J_p x \perp \zeta$ for all $x \in \zeta$. The condition is equivalent to $\omega|_\zeta = 0$. We call a three-dimensional submanifold M of S^6 a *Lagrangian submanifold* if the tangent space T_pM is a Lagrangian subspace at each point $p \in M$. The condition implies that ω restricted to M vanishes. Many researchers refer to Lagrangian submanifolds in S^6 as three-dimensional totally real submanifolds of S^6. The following remarkable theorem by N.Ejiri is known:

Theorem 3.2([2]) Any Lagrangian submanifold of S^6 is orientable and minimal.

We call an oriented Lagrangian subspace ζ a *special Lagrangian subspace* with respect to Ω if $\Omega(v_1, v_2, v_3) = 1$ for a positively oriented orthonormal basis $\{v_1, v_2, v_3\}$ of ζ. An oriented Lagrangian submanifold M is called a *special Lagrangian submanifold* if the tangent space T_pM is a special Lagrangian subspace at each point $p \in M$. From the view point of calibrated geometry, Mashimo showed the following:

Theorem 3.3([8]) If a three-dimensional oriented submanifold M of S^6 is Lagrangian, then it is a special Lagrangian submanifold, if necessary, we reverse its orientation.

We review basic facts on special Lagrangian subspaces of the tangent space of S^6.

Lemma 3.4 Let ζ be a three-dimensional oriented subspace of T_pS^6. Then either ζ or $-\zeta$ is special Lagrangian if and only if ζ is a Lagrangian subspace and $\varphi(\zeta) = 0$.

Lemma 3.5 Let ζ be a three-dimensional oriented subspace of T_pS^6. If ζ is special Lagrangian, $-J_p\zeta$ is associative. Conversely, if ζ is associative, $J_p\zeta$ is special Lagrangian.

§4 The two double fibrations

Nakata ([9]) showed the following double fibration motivated to construct a theory of Penrose type twistor correspondence for the geometries of S^6 and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Let $\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O})$ be a flag manifold defined by

$$\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O}) = \{(p,\xi) \in S^6 \times \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid p \in \xi \}.$$

We define maps $\bar{\omega}$ and π_- of $\widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O})$ onto S^6 and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ by the projections to the first factor and the second factor, respectively.

(4.1)
$$
\begin{array}{ccc}
 & \widetilde{\mathrm{Fl}}_{1,ass}(\mathrm{Im}\mathbb{O}) & \\
 \swarrow^{\bar{\omega}} & & \searrow^{\pi_-} \\
S^6 & & \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})
\end{array}
$$

These fibrations are G_2-equivariant. He proved the following interesting result:

Theorem 4.1([9]) In the double fibration (4.1), for each $p \in S^6$, $\pi_-(\bar{\omega}^{-1}(p))$ is a four-dimensional totally geodesic and quaternionic submanifold of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ which is isomorphic to $\mathbb{C}P^2$ and for each $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, $\bar{\omega}(\pi_-^{-1}(\xi))$ is a two-dimensional sphere S^2 which is totally geodesic and almost complex in S^6.

We consider another double fibration to associate the geometry of the associative Grassmann manifold with Lagrangian submanifolds of S^6. As a preparation of constructing the fibration, we recall the triple cross product:

$$x \times y \times z = \frac{1}{2}\left(x(\bar{y}z) - z(\bar{y}x)\right) \qquad \text{for } x, y, z \in \mathbb{O},$$

where $\bar{y}$ is the octonion conjugation for $y \in \mathbb{O}$. It is known that $\langle x, y \times z \times w\rangle$ are alternating in $x, y, z, w \in \mathbb{O}$ and that the real part of $x \times y \times z = \varphi(x, y, z)$ for $x, y, z \in \mathrm{Im}\mathbb{O}$. We recall the level set $\tilde{M}(0)$ defined in Section 2 :

$$\tilde{M}(0) = \{\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) \mid \varphi(\zeta) = 0\}.$$

Let $\{v_1, v_2, v_3\}$ be a positively oriented orthonormal basis of $\zeta \in \tilde{M}(0)$ and put $p = -v_1 \times v_2 \times v_3 = v_1(v_2v_3)$. Then since the real part of $v_1 \times v_2 \times v_3$ is equal to $\varphi(\zeta)$, we have $p \in \mathrm{Im}\mathbb{O}$. Moreover, it holds that $|p| = 1$ and p is orthogonal to ζ. Then ζ is a subspace of T_pS^6. Moreover ζ is a special Lagrangian subspace, hence by Lemma 3.5, $-J_p\zeta$ is associative. By these, we can define another double fibration:

(4.2)
$$
\begin{array}{ccc}
 & \tilde{M}(0) & \\
 \swarrow^{\chi} & & \searrow^{\pi} \\
S^6 & & \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}).
\end{array}
$$

$$\chi(\zeta) = -v_1 \times v_2 \times v_3 = p, \quad \pi(\zeta) = -J_p\zeta$$

These fibrations are also G_2-equivariant. For this double fibration, the similar result to Nakata' one holds:

Theorem 4.2 In the double fibration (4.2), for each $p \in S^6$, $\pi(\chi^{-1}(p))$ is a five-dimensional totally geodesic submanifold of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ which is isomorphic to $SU(3)/SO(3)$ and for each $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, $\chi(\pi^{-1}(\xi))$ is a three-dimensional sphere S^3 which is totally geodesic and special Lagrangian in S^6.

Remark. S.Klein gives the classification of the totally geodesic submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ in [6]. In the following table, we list the maximal totally geodesic submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.

maximal tot. geod.	dim.
$SU(3)/SO(3)$	5
$\mathbb{C}P^2$	4
$S^2_{r=1} \times S^2_{r=\frac{1}{\sqrt{3}}}/\mathbb{Z}_2$	4
$S^2_{r=\frac{2\sqrt{21}}{3}}$	2

Theorems 4.1 and 4.2 give the geometric realization of totally geodesic submanifolds $\mathbb{C}P^2$ and $SU(3)/SO(3)$.

It is important and useful for the submanifolds geometry of Riemannian symmetric spaces to characterize the tangent spaces of totally geodesic submanifolds, so called, curvature-invariant subspaces, or Lie triple systems. For our cases, we have the following: For $p \in S^6$, we put

$$\mathfrak{S}_p = \pi_-(\bar{\omega}^{-1}(p)) = \mathbb{C}P^2, \quad \mathcal{L}_p = \pi(\chi^{-1}(p)) = SU(3)/SO(3).$$

Then for $\xi \in \mathfrak{S}_p$ and $\xi \in \mathcal{L}_p$, we have the following:

$$\begin{aligned} T_\xi \mathfrak{S}_p &= \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \gamma(p) = 0 \}, \\ T_\xi \mathcal{L}_p &= \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \langle \gamma(v), p \rangle = 0 \text{ for any } v \in \xi \}, \end{aligned}$$

under the identification of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ with $\mathrm{Hom}_{ass}(\xi, \xi^\perp)$. Here we note that in the case of $\mathbb{C}P^2$, $p \in \xi$ and in the case of $SU(3)/SO(3)$, $p \in \xi^\perp$. They give the geometric description of the Lie triple systems. Here we note that $\mathfrak{S}_p = \mathfrak{S}_{-p}$, $\mathcal{L}_p = \mathcal{L}_{-p}$.

For $\xi \in \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ and a one-dimensional subspace l of $\xi^\perp$, we put

$$\mathfrak{m}_l = \{ \gamma \in T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \mid \langle \gamma(v), l \rangle = 0 \text{ for any } v \in \xi \}.$$

Then $\mathfrak{m}_l$ is a Lie triple system and coincides with the tangent space of $\mathcal{L}_p$ at ξ, where $p \in l \cap S^6$. A five-dimensional subspace $\mathfrak{m}$ of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is called a *SL-type subspace* if there exists a one-dimensional subspace l of $\xi^\perp$ such that $\mathfrak{m} = \mathfrak{m}_l$. For later use, we prepare the following Lemma:

Lemma 4.3 Let $\mathfrak{p}$ be a three-dimensional subspace of $T_\xi \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. If there exists a SL-type subspace $\mathfrak{m}$ such that $\mathfrak{p} \subset \mathfrak{m}$, it is unique.

§5 The Gauss maps of submanifolds of S^6 to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$

We show remarkable relations between Lagrangian submanifolds of S^6 and the geometry of the associative Grassmann manifold.

Let $f : M \to S^6$ be a Lagrangian immersion of a three-dimensional manifold M into S^6. Then by Theorem 3.3, f is special Lagrangian. By Lemma 3.5, $-J_{f(p)} df(T_pM)$ is associative. We obtain a kind of Gauss map $\nu : M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ defined by $\nu(p) = -J_{f(p)} df(T_pM)$.

Theorem 5.1 If $f : M \to S^6$ is a Lagrangian immersion, then the Gauss map $\nu : M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ associated to f is harmonic.

It is a similar result as a famous formula by E. A. Ruh and J. Vilms ([10]).

Next we consider a reconstruction of Lagrangian immersions from maps to the associative Grassmann manifold. First we define a new class of maps to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Let $\bar{\nu}\colon M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ be a smooth map of a three-dimensional manifold M to $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. We call $\bar{\nu}$ an *inclusive map of SL-type* if there exists a SL-type subspace $\mathfrak{m}$ of $T_{\bar{\nu}(p)}\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ at each point $p \in M$ such that $d\bar{\nu}(T_pM) \subset \mathfrak{m}$. Then we have the fact that the Gauss map ν associated to a Lagrangian immersion $f\colon M \to S^6$ is an inclusive map of SL-type. Conversely, let $\nu\colon M \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ be an inclusive immersion of SL-type of a three-dimensional manifold M. Then taking non-trivial double covering $\eta\colon M' \to M$, if necessary, we obtain a map $f\colon M' \to S^6$ which satisfies $f(p) \in (\nu \circ \eta(p))^{\perp}$ and $d(\nu \circ \eta)(T_pM') \subset \mathfrak{m}_{\mathbb{R}f(p)}$ at each $p \in M'$ due to Lemma 4.3.

Proposition 5.2 If f is an immersion, then f is Lagrangian and $\nu \circ \eta$ is the Gauss map associated to f and hence harmonic with respect to the induced Riemannian metric by f.

§6 Examples $\cdots$ Homogeneous Lagrangian submanifolds

Mashimo ([8]) classified compact Lagrangian submanifolds of S^6 which are obtained as orbits of closed subgroups of G_2. That is, it is a totally geodesic spehere or it is congruent to one of four kinds of Lagrangian submanifolds M_i $(i = 1, 2, 3, 4)$. They are orbits of three-dimensional Lie subgroups $SU(2)$ or $SO(3)$. The following is easy to prove:

Proposition 6.1 The Gauss map associated to a totally geodesic and Lagrangian submanifold is constant.

The Gauss maps $\nu_i\colon M_i \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ associated to M_i $(i = 1, \ldots, 4)$ are equivariant under the corresponding Lie groups. In this note, we explain M_3 and M_4 only. The following description of the action by $SO(3)$ is due to J. D. Lotay ([7]): We identify $\mathrm{Im}\mathbb{O}$ with the space $\mathcal{H}^3(\mathbb{R}^3)$ of homogeneous harmonic cubics on $\mathbb{R}^3$ by the following correspondence:

$$\begin{array}{ll} e_1 \mapsto \frac{\sqrt{10}}{10}x(2x^2 - 3y^2 - 3z^2); & \\ e_2 \mapsto -\sqrt{6}xyz; & e_3 \mapsto \frac{\sqrt{6}}{2}x(y^2 - z^2); \\ e_4 \mapsto -\frac{\sqrt{15}}{10}y(4x^2 - y^2 - z^2); & e_5 \mapsto -\frac{\sqrt{15}}{10}z(4x^2 - y^2 - z^2); \\ e_6 \mapsto \frac{1}{2}y(y^2 - 3z^2); & e_7 \mapsto -\frac{1}{2}z(z^2 - 3y^2). \end{array}$$

Then the standard $SO(3)$ action on $\mathbb{R}^3$ induces an action on $\mathcal{H}^3(\mathbb{R}^3)$. Then by this action $SO(3)$ is a subgroup of G_2. Let M_3 and M_4 be the orbits through e_2 and e_6 of this $SO(3)$-action, respectively. Then $M_3 = SO(3)/A_4$ and $M_4 = SO(3)/S_3$ are Lagrangian submanifolds of S^6. In particular, M_3 is of constant curvature $\frac{1}{16}$.

We compute the differentials $d\nu_i\colon T_xM_i \to T_{\nu_i(x)}\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ of the Gauss maps at $x \in M_i$ $(i = 3, 4)$, respectively, and determine the ranks of $d\nu_i$.

Proposition 6.2 The rank of $d\nu_3$ is equal to 3 and the rank of $d\nu_4$ is equal to 2.

Especially in the case of M_3, the following holds:

Proposition 6.3 The Gauss map $\nu_3\colon M_3 \to \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is a minimal immersion with respect to $\frac{15}{8}\langle , \rangle$, where $\langle , \rangle$ is the metric on M_3.

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